

New realizations of the supergroup $D(2, 1; \alpha)$ in $\mathcal{N} = 4$ superconformal mechanics

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Abstract

We present new explicit realizations of the most general $\mathcal{N} = 4, d = 1$ superconformal symmetry $D(2, 1; \alpha)$ in the models of $\mathcal{N} = 4$ superconformal mechanics based on the reducible multiplets $(\mathbf{1}, \mathbf{4}, \mathbf{3}) \oplus (\mathbf{0}, \mathbf{4}, \mathbf{4})$, $(\mathbf{3}, \mathbf{4}, \mathbf{1}) \oplus (\mathbf{0}, \mathbf{4}, \mathbf{4})$ and $(\mathbf{4}, \mathbf{4}, \mathbf{0}) \oplus (\mathbf{0}, \mathbf{4}, \mathbf{4})$. We start from the manifestly supersymmetric superfield actions for these systems and then descend to the relevant off- and on-shell component actions from which we derive the $D(2, 1; \alpha)$ (super)charges by the Noether procedure. Some peculiarities of these realizations of $D(2, 1; \alpha)$ are discussed. We also construct a new $D(2, 1; \alpha)$ invariant system by joining the multiplets $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ and $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ in such a way that they interact with each other through an extra $(\mathbf{0}, \mathbf{4}, \mathbf{4})$ multiplet. New fermionic conformal couplings appear as the result of elimination of the appropriate auxiliary fields.

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1 Introduction

Superconformal mechanics (SCM) [1] - [3] has plenty of applications [4] - [28]. For instance, the SCM models can be identified with the denominator theories in the $\text{AdS}_2/\text{CFT}_1$ version of the general AdS/CFT correspondence and used for the microscopic description of the extremal black holes [4]-[8], [14], [27]. Various versions of the $\mathcal{N} = 4$ SCM are of special interest in these respects, since they provide an explicit description of the massive $\mathcal{N} = 4$ superparticle moving near the horizon of an extreme Reissner-Nordström black hole (see, e.g., [4], [6], [8], [27]). An important class of the multiparticle SCM models is constituted by integrable superconformal Calogero-type systems [6], [13], [15], [16], [21]. A review of possible implications of SCM in various domains, including $\mathcal{N} = 4$ case, and additional references can be found in [23].

The most general $\mathcal{N} = 4, d = 1$ superconformal group is the exceptional supergroup $D(2, 1; \alpha)$ [29], [30]. At $\alpha = 0, -1$ it reduces to the semi-direct product $PSU(1, 1|2) \ltimes SU(2)$ and at $\alpha = -\frac{1}{2}$ to the supergroup $OSp(4|2)$ ¹. The realizations of $D(2, 1; \alpha)$ in the models of supersymmetric mechanics were a subjects of many works (see, e.g., [7]-[12], [17]-[22], [24], [25], [28], [23] and references therein)². As a rule, the realizations on one or another fixed type of the irreducible $\mathcal{N} = 4, d = 1$ supermultiplet were considered. Recently, the study of $SU(1, 1|2)$ superconformal systems including some pairs of such multiplets was initiated in Ref. [27]. Some interesting links with the $\mathcal{N} = 2, d = 5$ supergravity were established there. One of the basic points of the construction in [27] was the inclusion of couplings with the fermionic $\mathcal{N} = 4$ multiplets $(\mathbf{0}, \mathbf{4}, \mathbf{4})$ which do not enlarge the dimension of the target bosonic manifold.

In the present paper we study the analogous realizations of $D(2, 1; \alpha)$ as distinct from [27], where the particular $SU(1, 1|2)$ case was treated. Another new point of our consideration is that in all cases we start from the manifestly $\mathcal{N} = 4$ supersymmetric off-shell superfield description of the relevant multiplet pairs and write down the off-shell component Lagrangians, while in [27] only on-shell versions of the latter were addressed. Keeping the relevant auxiliary fields in the combined actions of different pairs allows one to get more general on-shell component actions after elimination of these fields by their algebraic equations of motion.³

Since the natural off-shell description of the multiplets considered in [27] and in the present paper is achieved in the framework of $\mathcal{N} = 4, d = 1$ harmonic superspace [32], we start in section 2 with recalling the basics of this approach. Then, in section 3, we present the superfield and component descriptions of the fermionic multiplet $(\mathbf{0}, \mathbf{4}, \mathbf{4})$, which is the common part of all systems considered in [27] and here. In section 4 we describe the $D(2, 1; \alpha)$ invariant system of interacting $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ and $(\mathbf{0}, \mathbf{4}, \mathbf{4})$ multiplets, both in the superfield and the component formulations, and give the precise form of the $D(2, 1; \alpha)$ generators for this case. In sections 5 and 6 we do the same for the multiplet pair $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ and $(\mathbf{0}, \mathbf{4}, \mathbf{4})$, as well as for the pair $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{4}, \mathbf{4})$. In section 7, as an example of the power of the off-shell approach, we present a new superconformal system involving the triple of the multiplets $(\mathbf{1}, \mathbf{4}, \mathbf{3})$, $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{4}, \mathbf{4})$. The elimination of the auxiliary fields in the corresponding

¹The isomorphic superalgebras are related by the redefinition $\alpha \rightarrow -(1 + \alpha)$.

²For implications of $D(2, 1; \alpha)$ in string theory and AdS/CFT correspondence see, e.g., [31].

³As distinct from the $D(2, 1; \alpha)$ invariant systems with the so-called $\mathcal{N} = 4$ spin multiplets [18], [20], in our case all bosonic fields of physical dimension are dynamical.

action yields new fermionic terms which are absent in the actions of the relevant isolated pairs. Section 8 contains conclusions and outlook.

2 $\mathcal{N} = 4, d = 1$ harmonic superspace

The harmonic analytic $\mathcal{N}=4, d = 1$ superspace [32, 33, 34] as the one-dimensional version of the general harmonic superspace [35] is defined as the following coordinate set

$$(\zeta, u) = (t_A, \theta^+, \bar{\theta}^+, u_i^\pm), \quad u^{+i} u_i^- = 1. \quad (2.1)$$

These coordinates are related to the standard $\mathcal{N}=4, d=1$ superspace (central basis) coordinates $z = (t, \theta_i, \bar{\theta}^i)$, $(\bar{\theta}_i) = \bar{\theta}^i$ as

$$t_A = t + i(\theta^+ \bar{\theta}^- + \theta^- \bar{\theta}^+), \quad \theta^\pm = \theta^i u_i^\pm, \quad \bar{\theta}^\pm = \bar{\theta}^i u_i^\pm. \quad (2.2)$$

The $\mathcal{N}=4$ covariant spinor derivatives and their harmonic projections are defined by

$$D^i = \frac{\partial}{\partial \theta_i} - i \bar{\theta}^i \partial_t, \quad \bar{D}_i = \frac{\partial}{\partial \bar{\theta}^i} - i \theta_i \partial_t, \quad \overline{(D^i)} = -\bar{D}_i, \quad \{D^i, \bar{D}_k\} = -2i \delta_k^i \partial_t, \quad (2.3)$$

$$D^\pm = u_i^\pm D^i, \quad \bar{D}^\pm = u_i^\pm \bar{D}^i, \quad \{D^+, \bar{D}^-\} = -\{D^-, \bar{D}^+\} = -2i \partial_{t_A}. \quad (2.4)$$

In the analytic basis $z_A = (t_A, \theta^\pm, \bar{\theta}^\pm, u^\pm)$, the derivatives D^+ and \bar{D}^+ are short,

$$D^+ = \frac{\partial}{\partial \theta^-}, \quad \bar{D}^+ = -\frac{\partial}{\partial \bar{\theta}^-}. \quad (2.5)$$

The analyticity-preserving harmonic derivative D^{++} and its conjugate D^{--} are given by

$$\begin{aligned} D^{++} &= \partial^{++} + 2i\theta^+ \bar{\theta}^+ \partial_{t_A} + \theta^+ \frac{\partial}{\partial \theta^-} + \bar{\theta}^+ \frac{\partial}{\partial \bar{\theta}^-}, \\ D^{--} &= \partial^{--} + 2i\theta^- \bar{\theta}^- \partial_{t_A} + \theta^- \frac{\partial}{\partial \theta^+} + \bar{\theta}^- \frac{\partial}{\partial \bar{\theta}^+}, \quad \partial^{\pm\pm} = u_i^\pm \frac{\partial}{\partial u_i^\mp}, \end{aligned} \quad (2.6)$$

and become the pure partial derivatives $\partial^{\pm\pm}$ in the central basis. They satisfy the relations

$$[D^{++}, D^{--}] = D^0, \quad [D^0, D^{\pm\pm}] = \pm 2D^{\pm\pm}, \quad (2.7)$$

where D^0 is the operator counting external harmonic $U(1)$ charges. The integration measures in the full harmonic superspace and its analytic subspace are defined as

$$\begin{aligned} \mu_H &= dudt d^4\theta = dudt_A (D^- \bar{D}^-) (D^+ \bar{D}^+) = \mu_A^{(-2)} (D^+ \bar{D}^+), \\ \mu_A^{(-2)} &= dud\zeta^{(-2)} = dudt_A d\theta^+ d\bar{\theta}^+ = dudt_A (D^- \bar{D}^-). \end{aligned} \quad (2.8)$$

The analytic subspace (ζ, u) is closed under the action of the most general $\mathcal{N} = 4, d = 1$ superconformal group $D(2, 1; \alpha)$ and its degenerate $D(2, 1; \alpha = 0)$ and $D(2, 1; \alpha = -1)$ cases which are reduced to the semi-direct products $PSU(1, 1|2) \ltimes SU(2)_{\text{ext}}$. In what follows, we will need the transformation properties of some relevant quantities under the ‘‘Poincaré’’ and conformal supersymmetry. The invariance under these transformations is sufficient for

ensuring the complete $D(2, 1; \alpha)$ invariance since the rest of the $D(2, 1; \alpha)$ transformations is contained in the closure of the conformal and manifest Poincaré $\mathcal{N}=4, d=1$ supersymmetries.

In the $\mathcal{N}=4$ superfield approach, the invariance under the ordinary $d = 1$ supertranslations ($\bar{\varepsilon}^i = \overline{(\varepsilon_i)}$)

$$\delta t = i(\varepsilon_k \bar{\theta}^k - \theta_k \bar{\varepsilon}^k), \quad \delta \theta_k = \varepsilon_k, \quad \delta \bar{\theta}^k = \bar{\varepsilon}^k \quad (2.9)$$

and

$$\delta t_A = 2i(\varepsilon^- \bar{\theta}^+ - \bar{\varepsilon}^- \theta^+), \quad \delta \theta^+ = \varepsilon^+, \quad \delta \bar{\theta}^+ = \bar{\varepsilon}^+, \quad (2.10)$$

where $\varepsilon^\pm = \varepsilon^i u_i^\pm$, $\bar{\varepsilon}^\pm = \bar{\varepsilon}^i u_i^\pm$, is automatic.

The coordinate realization of the superconformal $D(2, 1; \alpha)$ boosts is as follows :

$$\delta' t = it(\theta_k \bar{\eta}^k + \bar{\theta}^k \eta_k) - (1 + \alpha) \theta_i \bar{\theta}^i (\theta_k \bar{\eta}^k + \bar{\theta}^k \eta_k), \quad (2.11)$$

$$\delta' \theta_i = -\eta_i t - 2i\alpha \theta_i (\theta_k \bar{\eta}^k) + 2i(1 + \alpha) \theta_i (\bar{\theta}^k \eta_k) - i(1 + 2\alpha) \eta_i (\theta_k \bar{\theta}^k), \quad (2.12)$$

$$\delta' \bar{\theta}^i = -\bar{\eta}^i t - 2i\alpha \bar{\theta}^i (\bar{\theta}^k \eta_k) + 2i(1 + \alpha) \bar{\theta}^i (\theta_k \bar{\eta}^k) + i(1 + 2\alpha) \bar{\eta}^i (\theta_k \bar{\theta}^k), \quad (2.13)$$

$$\delta' t_A = \alpha^{-1} \Lambda_{sc} t_A, \quad \delta' u_i^+ = \Lambda^{++} u_i^-, \quad (2.14)$$

$$\delta' \theta^+ = -\eta^+ t_A + 2i(1 + \alpha) \eta^- \theta^+ \bar{\theta}^+, \quad \delta' \bar{\theta}^+ = -\bar{\eta}^+ t_A + 2i(1 + \alpha) \bar{\eta}^- \theta^+ \bar{\theta}^+, \quad (2.15)$$

$$\delta' (dtd^4\theta) = -(dtd^4\theta) \Lambda_0, \quad \delta' \mu_H = \mu_H (2\Lambda_{sc} - (1 + \alpha) \Lambda_0), \quad \delta' \mu_A^{(-2)} = 0, \quad (2.16)$$

$$\delta' \mu_A^{(-2)} = 0, \quad \delta' du = du D^{--} \Lambda^{++}, \quad (2.17)$$

$$\delta' D^{++} = -\Lambda^{++} D^0, \quad \delta' D^0 = 0. \quad (2.18)$$

Here $\eta^\pm = \eta^i u_i^\pm$, $\bar{\eta}^\pm = \bar{\eta}^i u_i^\pm$, $\bar{\eta}^i = \overline{(\eta_i)}$, and

$$\Lambda_{sc} = 2i\alpha(\bar{\eta}^- \theta^+ - \eta^- \bar{\theta}^+), \quad \Lambda^{++} = D^{++} \Lambda_{sc} = 2i\alpha(\bar{\eta}^+ \theta^+ - \eta^+ \bar{\theta}^+), \quad D^{++} \Lambda^{++} = 0, \quad (2.19)$$

$$\Lambda_0 = \alpha^{-1} (2\Lambda_{sc} - D^{--} \Lambda^{++}) = 2i(\theta_k \bar{\eta}^k + \bar{\theta}^k \eta_k), \quad D^{++} \Lambda_0 = 0. \quad (2.20)$$

The symbol \sim means the generalized tilde-conjugation [35]. With such definitions, all the coordinate transformations contain no singularities in the degenerate $\alpha = 0$ or $\alpha = -1$ cases.

3 The fermionic multiplet $(0, 4, 4)$

This multiplet is the fermionic analog of the multiplet $(4, 4, 0)$. It is described off shell by the fermionic analytic superfield Ψ^{+A} , $A = 1, 2$, $(\widetilde{\Psi^{+A}}) = \Psi_A^+$, satisfying the constraint [32]:

$$D^{++} \Psi^{+A} = 0 \quad \Rightarrow \quad \Psi^{+A} = \phi^{iA} u_i^+ + \theta^+ F^A + \bar{\theta}^+ \bar{F}^A - 2i\theta^+ \bar{\theta}^+ \dot{\phi}^{iA} u_i^-, \quad (3.1)$$

where $(\overline{\phi^{iA}}) = -\phi_{iA}$, $(\overline{F^A}) = \bar{F}_A$. On the index A , the appropriate $SU(2)_{PG}$ group acts. Its generators commute with the $\mathcal{N} = 4$ supersymmetry and $D(2, 1; \alpha)$ generators. The requirement of superconformal covariance of the constraint (3.1) uniquely fixes the superconformal $D(2, 1; \alpha)$ transformation rule of Ψ^{+A} , for any α , as

$$\delta_{sc} \Psi^{+A} = \Lambda_{sc} \Psi^{+A}. \quad (3.2)$$

Off-shell transformations of component fields are the following

$$\begin{aligned}\delta\phi^{iA} &= -(\omega^i F^A + \bar{\omega}^i \bar{F}^A), \\ \delta F^A &= 2i\bar{\omega}^k \dot{\phi}_k^A + 2i\alpha \bar{\eta}^k \phi_k^A, \quad \delta \bar{F}_A = 2i\omega_k \dot{\phi}_A^k + 2i\alpha \eta_k \bar{\phi}_A^k,\end{aligned}\tag{3.3}$$

where

$$\omega_i = \varepsilon_i - t\eta_i, \quad \bar{\omega}^i = \bar{\varepsilon}^i - t\bar{\eta}^i.$$

In the central basis, the constraint (3.1) and the analyticity conditions $D^+\Psi^{+A} = \bar{D}^+\Psi^{+A} = 0$ imply

$$\Psi^{+A}(z, u) = \Psi^{iA}(z)u_i^+, \tag{3.4}$$

$$D^{(i}\Psi^{k)A}(z) = \bar{D}^{(i}\Psi^{k)A}(z) = 0. \tag{3.5}$$

The free action of Ψ^{+A} ,

$$S_{free}^{(\Psi)} = \frac{1}{2} \int dud\zeta^{(-2)} \Psi^{+A} \Psi_A^+ = \int dt \left(-i\phi^{iA} \dot{\phi}_{iA} + F^A \bar{F}_A \right), \tag{3.6}$$

is not invariant under $D(2, 1; \alpha)$, except for the special case of $\alpha=0$, in which we will not be too interested. As we will see, the superconformal versions of the free Ψ^{+A} action, which are valid for any α , can be constructed by coupling this multiplet to those considered in the next sections.

The only additional $\mathcal{N}=4$ invariant is the appropriate Fayet-Iliopoulos (FI)-type term

$$S_{FI}^{(\Psi)} = \gamma \int dud\zeta^{(-2)} (\theta^+ \xi_A \Psi^{+A} + \bar{\theta}^+ \bar{\xi}^A \Psi_A^+) = \gamma \int dt (\bar{\xi}_A F^A - \xi^A \bar{F}_A), \tag{3.7}$$

$\xi_A, \bar{\xi}^A = (\bar{\xi}_A)$ being $SU(2)_{PG}$ breaking constants. It is superconformal at $\alpha = -1$ [34].

4 The multiplet pair $(1, 4, 3) \oplus (0, 4, 4)$

4.1 The multiplet $(1, 4, 3)$

The off-shell multiplet $(1, 4, 3)$ is described by a real $\mathcal{N}=4$ superfield $X(z)$ obeying the constraints [3]

$$D^i D_i X = \bar{D}^i \bar{D}_i X = 0, \quad [D^i, \bar{D}_i] X = 0. \tag{4.1}$$

Solution of these constraints is provided by

$$X(t, \theta_i, \bar{\theta}^i) = r + \theta_i \varphi^i + \bar{\varphi}_i \bar{\theta}^i + i\theta^i \bar{\theta}^k A_{ik} - \frac{i}{2}(\theta)^2 \dot{\varphi}_i \bar{\theta}^i - \frac{i}{2}(\bar{\theta})^2 \theta_i \dot{\bar{\varphi}}^i + \frac{1}{4}(\theta)^2 (\bar{\theta})^2 \ddot{r}, \tag{4.2}$$

where $(\bar{r}) = r$, $(\bar{\varphi}^i) = \bar{\varphi}_i$, $(\bar{A}^{ik}) = A_{ik} = A_{(ik)}$ and $(\theta)^2 = \theta_k \theta^k$, $(\bar{\theta})^2 = \bar{\theta}^k \bar{\theta}_k$.

The same constraints (4.1) rewritten in harmonic superspace read

$$D^{++}X = 0, \quad D^+ D^- X = \bar{D}^+ \bar{D}^- X = 0, \quad (D^+ \bar{D}^- + \bar{D}^+ D^-) X = 0. \tag{4.3}$$

The extra harmonic constraint guarantees the harmonic independence of X in the central basis.

As was shown in [34], this multiplet can be also described in terms of the real analytic gauge superfield $\mathcal{V}(\zeta, u)$ with the abelian gauge transformation

$$\mathcal{V} \Rightarrow \mathcal{V}' = \mathcal{V} + D^{++}\Lambda^{--}, \quad \Lambda^{--} = \Lambda^{--}(\zeta, u). \quad (4.4)$$

In the Wess-Zumino (WZ) gauge just the irreducible $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ content remains

$$\mathcal{V}_{WZ}(\zeta, u) = r(t_A) - 2\theta^+\varphi^i(t_A)u_i^- - 2\bar{\theta}^+\bar{\varphi}^i(t_A)u_i^- + 3i\theta^+\bar{\theta}^+A^{(ik)}(t_A)u_i^-u_k^-. \quad (4.5)$$

No residual gauge freedom is left. The original superfield $X(z)$ is related to $\mathcal{V}(\zeta, u)$ by

$$X(t, \theta^i, \bar{\theta}_k) = \int du \mathcal{V} \left(t + 2i\theta^i\bar{\theta}^k u_{(i}^+ u_{k)}^-, \theta^i u_i^+, \bar{\theta}^k u_k^+, u_l^\pm \right). \quad (4.6)$$

The constraints (4.1) are recovered as a consequence of the harmonic analyticity of \mathcal{V} ,

$$D^+\mathcal{V} = \bar{D}^+\mathcal{V} = 0. \quad (4.7)$$

The transformation properties of the superfields X and \mathcal{V} under the $D(2, 1; \alpha)$ conformal supersymmetry are defined by

$$\delta_{sc} X = -\alpha \Lambda_0 X, \quad \delta_{sc} \mathcal{V} = -2\Lambda_{sc} \mathcal{V}. \quad (4.8)$$

The full set of the component fermionic transformations obtained from (2.9) - (2.16) reads

$$\begin{aligned} \delta r &= -\omega_i \varphi^i + \bar{\omega}^i \bar{\varphi}_i, \\ \delta \varphi^i &= i\bar{\omega}^i \dot{r} - i\bar{\omega}_k A^{ki} - 2i\alpha \bar{\eta}^i r, \quad \delta \bar{\varphi}_i = -i\omega_i \dot{r} - i\omega^k A_{ki} + 2i\alpha \eta_i r, \\ \delta A_{ik} &= -2(\omega_{(i} \dot{\varphi}_{k)} + \bar{\omega}_{(i} \dot{\bar{\varphi}}_{k)}) + 2(1 + 2\alpha)(\eta_{(i} \varphi_{k)} + \bar{\eta}_{(i} \bar{\varphi}_{k)}), \end{aligned} \quad (4.9)$$

where $\omega_i = \varepsilon_i - t\eta_i$ and $\bar{\omega}^i = \bar{\varepsilon}^i - t\bar{\eta}^i$.

The general $\varepsilon_i, \bar{\varepsilon}_k$ -invariant superfield action of the multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ is written as

$$S_{gen}^{(X)} = \int dt d^4\theta \mathcal{L}_{gen}(X). \quad (4.10)$$

The action invariant under $D(2, 1; \alpha)$ (except for the special value of $\alpha=0$) is [11]

$$S_{sc}^{(X)} = -\frac{1}{8(1+\alpha)} \int dt d^4\theta (X^{-1/\alpha} - X). \quad (4.11)$$

For the correct $d=1$ field theory interpretation, one must assume that X develops a non-zero background value, $X = 1 + \dots$. Note that $\int dt d^4\theta X$ is an integral of total derivative and does not contribute for $\alpha \neq -1$. We add this term for ensuring the correct limit $\alpha = -1$, in which (4.11) is reduced to

$$S_{sc}^{(X)(\alpha=-1)} = -\frac{1}{8} \int dt d^4\theta X \log X. \quad (4.12)$$

Using the coordinate and superfield transformations (2.11) - (2.17), (4.8), it is easy to check the $D(2, 1; \alpha)$ invariance of the action (4.11) and the covariance of the relation (4.6).

In the component field notation the action (4.11) takes the form

$$\begin{aligned}
S_{sc}^{(X)} &= \frac{1}{8\alpha^2} \int dt r^{-\frac{1}{\alpha}-2} \left[\dot{r}\dot{r} + i \left(\bar{\varphi}_k \dot{\varphi}^k - \dot{\bar{\varphi}}_k \varphi^k \right) + \frac{1}{2} A^{ik} A_{ik} \right] \\
&\quad - \frac{i}{8\alpha^2} \left(\frac{1}{\alpha} + 2 \right) \int dt r^{-\frac{1}{\alpha}-3} A^{ik} \varphi_{(i} \bar{\varphi}_{k)} \\
&\quad - \frac{1}{24\alpha^2} \left(\frac{1}{\alpha} + 2 \right) \left(\frac{1}{\alpha} + 3 \right) \int dt r^{-\frac{1}{\alpha}-4} \varphi^i \bar{\varphi}^k \varphi_{(i} \bar{\varphi}_{k)}.
\end{aligned} \tag{4.13}$$

One can also construct an $\mathcal{N}=4$ supersymmetric FI term

$$S_{FI}^{(X)} = i \int du d\zeta^{(-2)} c^{+2} \mathcal{V}, \quad c^{+2} = c^{ik} u_i^+ u_k^+, \quad [c] = cm^{-1}. \tag{4.14}$$

It generates a scalar potential after elimination of the auxiliary field $A^{(ik)}$ in the sum of (4.10) and (4.14). The FI term is superconformal only for the special choice $\alpha=0$ [34]. At this value of α , the corresponding scalar potential is the standard inverse-square conformal potential $\sim c^2 = c^{ik} c_{ik}$. The $\alpha = 0$ conformal kinetic sigma-model term can be constructed at cost of modifying the superconformal transformation law of X for this value of α as

$$\delta_{sc}^{(\alpha=0)} X = -\Lambda_0, \quad \delta_{sc}^{(\alpha=0)} \mathcal{V} = 4i \left(\bar{\varepsilon}^- \theta^+ - \varepsilon^- \bar{\theta}^+ \right), \tag{4.15}$$

under which the constraints (4.1) are still covariant. Then the superconformal kinetic term is given by

$$S_{sc}^{(X)(\alpha=0)} = \int dt d^4\theta e^X. \tag{4.16}$$

The FI term (4.14) remains invariant under the modified transformation rule (4.15).

For the choice of $\alpha = -1$, the conformal component potential can be secured by modifying the last constraint in (4.1) as

$$[D^i, \bar{D}_i] X = 0 \Rightarrow [D^i, \bar{D}_i] X = c, \tag{4.17}$$

where c is a real constant. At $\alpha = -1$ the modified constraints are still superconformally covariant. The conformal potential appears with the strength $\sim c^2$. Note that the superfield X subjected to the modified constraints is related to $X_{(c=0)}$ as

$$X = X_{(c=0)} + \frac{1}{2} c \bar{\theta}^i \theta_i = X_{(c=0)} + \frac{1}{2} c \left(\bar{\theta}^+ \theta^- - \bar{\theta}^- \theta^+ \right). \tag{4.18}$$

For $v_{(c=0)}$ there is still valid the prepotential representation (4.6). For X to have the correct $\alpha = -1$ transformation law, $\delta_{sc}^{(\alpha=-1)} X = \Lambda_0 X$, the transformation rule of the prepotential \mathcal{V} under the Poincaré and conformal supersymmetries should be modified as

$$\delta_{sc}^{(\alpha=-1)} \mathcal{V} = -2\Lambda_{sc}^{(\alpha=-1)} \mathcal{V} + c \left[(\epsilon^- + t_A \varepsilon^-) \bar{\theta}^+ + (\bar{\epsilon}^- + t_A \bar{\varepsilon}^-) \theta^+ \right]. \tag{4.19}$$

4.2 Superconformal coupling of multiplets (1, 4, 3) and (0, 4, 4)

Using the description of the multiplet (1, 4, 3) through the analytic prepotential \mathcal{V} , it is easy to construct its superconformal coupling to Ψ^{+A} [34]

$$S_{sc}^{(X,\Psi)} = \frac{1}{2} b \int dud\zeta^{(-2)} \mathcal{V} \Psi^{+A} \Psi_A^+. \quad (4.20)$$

This action is superconformal at any $\alpha \neq 0^4$ and it also respects the gauge invariance (4.4) as a consequence of the constraint (3.1). Assuming that $\mathcal{V} = 1 + \tilde{\mathcal{V}}$, $v = 1 + \tilde{v}$, (4.20) can be treated as a superconformal generalization of the free action (3.6). An analysis based on dimensionality and on the Grassmann character of the superfields Ψ^{+A} , $\Psi^{-A} = D^{--}\Psi^{+A}$ shows that the coupling (4.20) is the only possible coupling of this fermionic multiplet to the multiplet (1, 4, 3) preserving the canonical number of time derivatives in the component action (no more than two for bosons and no more than one for fermions).

It is easy to obtain the component-field representation of (4.20)

$$S_{sc}^{(X,\Psi)} = b \int dt \left[r \left(-i\phi^{iA} \dot{\phi}_{iA} + F^A \bar{F}_A \right) + \frac{i}{2} A^{ik} \phi_i^A \phi_{kA} + \bar{\varphi}^k \phi_{kA} F^A - \varphi_k \phi^{kA} \bar{F}_A \right]. \quad (4.21)$$

After summing up this action with the component superconformal (1, 4, 3) action (4.13) and eliminating the auxiliary fields A_{ik} and F^A, \bar{F}_A ,

$$\begin{aligned} A_{ik} &= i \left(\frac{1}{\alpha} + 2 \right) r^{-1} \varphi_{(i} \bar{\varphi}_{k)} - 4b\alpha^2 i r^{\frac{1}{\alpha}+2} \phi_{(i}^A \phi_{k)A}, \\ F^A &= r^{-1} \varphi_k \phi^{kA}, \quad \bar{F}_A = -r^{-1} \bar{\varphi}^k \phi_{kA}, \end{aligned} \quad (4.22)$$

we obtain the total on-shell superconformal action as

$$\begin{aligned} S_{(X+\Psi)} &= \frac{1}{8\alpha^2} \int dt r^{-\frac{1}{\alpha}-2} \left[\dot{r}\dot{r} + i \left(\bar{\varphi}_k \dot{\varphi}^k - \dot{\bar{\varphi}}_k \varphi^k \right) \right] - ib \int dt r \phi^{iA} \dot{\phi}_{iA} \\ &+ \int dt \left[\frac{1+2\alpha}{48\alpha^4} r^{-\frac{1}{\alpha}-4} \varphi_{(i} \bar{\varphi}_{k)} \varphi^i \bar{\varphi}^k - \frac{b}{2\alpha} r^{-1} \varphi_{(i} \bar{\varphi}_{k)} \phi^{iA} \phi_A^k \right. \\ &\quad \left. + \alpha^2 b^2 r^{\frac{1}{\alpha}+2} \phi_{(i}^A \phi_{k)A} \phi^{iB} \phi_B^k \right]. \end{aligned} \quad (4.23)$$

Redefining the field variables as

$$x = r^{-\frac{1}{2\alpha}}, \quad \psi_k = -\frac{1}{2\alpha} r^{-\frac{1}{2\alpha}-1} \varphi_k, \quad \chi^{iA} = \sqrt{2br} \phi^{iA}, \quad (4.24)$$

we cast the action (4.23) into the convenient form

$$\begin{aligned} S_{(X+\Psi)} &= \frac{1}{2} \int dt \left[\dot{x}\dot{x} + i \left(\bar{\psi}_k \dot{\psi}^k - \dot{\bar{\psi}}_k \psi^k \right) - i \chi^{iA} \dot{\chi}_{iA} \right] \\ &+ \int dt \frac{1}{x^2} \left[\frac{1}{3} (1 + 2\alpha) \psi_{(i} \bar{\psi}_{k)} \psi^i \bar{\psi}^k - \alpha \psi_{(i} \bar{\psi}_{k)} \chi^{iA} \chi_A^k \right. \\ &\quad \left. + \frac{1}{4} \alpha^2 \chi_{(i}^A \chi_{k)A} \chi^{iB} \chi_B^k \right]. \end{aligned} \quad (4.25)$$

⁴It is not invariant under the modified $\alpha = 0$ transformation of \mathcal{V} , eq. (4.15), as well as under (4.19) corresponding to the $c \neq 0, \alpha = -1$ version of (1, 4, 3) multiplet. So these options are excluded.

It is invariant under the following on-shell supersymmetry transformations

$$\delta x = -\omega_i \psi^i + \bar{\omega}^i \bar{\psi}_i, \quad (4.26)$$

$$\delta \psi^i = i\bar{\omega}^i \dot{x} + i\bar{\eta}^i x - (1+2\alpha) \frac{\omega_k \psi^k \psi^i + \bar{\omega}_k \psi^k \bar{\psi}^i}{x} + \alpha \frac{\bar{\omega}_k \chi^{iA} \chi_A^k}{x}, \quad (4.27)$$

$$\delta \bar{\psi}_i = -i\omega_i \dot{x} - i\eta_i x + (1+2\alpha) \frac{\bar{\omega}^k \bar{\psi}_k \bar{\psi}_i + \omega^k \bar{\psi}_k \psi_i}{x} + \alpha \frac{\omega^k \chi_{(i} \chi_{k)A}}{x}, \quad (4.28)$$

$$\delta \chi^{iA} = -2\alpha \frac{\omega^{(i} \psi^{k)} + \bar{\omega}^{(i} \bar{\psi}^{k)}}{x} \chi_k^A. \quad (4.29)$$

From (4.29) we observe that the supersymmetry transformation of χ^{iA} looks as a field-dependent $SU(2)$ transformation. At $\alpha = -1$, the action (4.25) is reduced to the corresponding on-shell action from Ref. [27].

Using the component $D(2, 1; \alpha)$ transformations (4.26), (4.27), (4.28) and (4.29), one can find the related Noether (super)charges

$$Q^i = p \psi^i - ix^{-1} \psi_k \left[\frac{2}{3} (1+2\alpha) \psi^{(i} \bar{\psi}^{k)} - \alpha \chi^{iA} \chi_A^k \right], \quad (4.30)$$

$$\bar{Q}_i = p \bar{\psi}_i + ix^{-1} \bar{\psi}^k \left[\frac{2}{3} (1+2\alpha) \psi_{(i} \bar{\psi}_{k)} - \alpha \chi_i^A \chi_{kA} \right] \quad (4.31)$$

and

$$S^i = x \psi^i - t Q^i, \quad \bar{S}_i = x \bar{\psi}_i - t \bar{Q}_i, \quad (4.32)$$

where $p \equiv \dot{x}$, and then check that they generate the classical $D(2, 1; \alpha)$ superalgebra.

To this end, we define the non-vanishing canonical Dirac brackets (at equal times) as

$$\{x, p\}_D = 1, \quad \{\psi^i, \bar{\psi}_j\}_D = -i \delta_j^i, \quad \{\chi^{iA}, \chi^{jB}\}_D = i \epsilon^{ij} \epsilon^{AB}, \quad (4.33)$$

where we adopted the convention $\epsilon_{12} = \epsilon^{21} = 1$. Using (4.33), we arrive at the following Dirac brackets for fermionic generators

$$\begin{aligned} \{Q^i, \bar{Q}_k\}_D &= -2i \delta_k^i H, & \{Q^i, Q^k\}_D &= 0, & \{\bar{Q}_i, \bar{Q}_k\}_D &= 0, \\ \{S^i, \bar{S}_k\}_D &= -2i \delta_k^i K, & \{S^i, S^k\}_D &= 0, & \{\bar{S}_i, \bar{S}_k\}_D &= 0, \\ \{Q^i, S^k\}_D &= 2i(1+\alpha) \epsilon^{ik} I, & \{\bar{Q}_i, \bar{S}_k\}_D &= -2i(1+\alpha) \epsilon_{ik} \bar{I}, \\ \{Q^i, \bar{S}_k\}_D &= 2i \delta_k^i D + 2i\alpha J_k^i - 2i(1+\alpha) \delta_k^i I_3, \\ \{\bar{Q}_i, S^k\}_D &= 2i \delta_i^k D - 2i\alpha J_i^k + 2i(1+\alpha) \delta_i^k I_3. \end{aligned} \quad (4.34)$$

Here, the bosonic generators

$$H = \frac{1}{2} p^2 + x^{-2} \left[\frac{1}{4} (1+2\alpha) \psi_i \psi^i \bar{\psi}_k \bar{\psi}^k + \alpha \psi_i \bar{\psi}_k \chi^{iA} \chi_A^k - \frac{1}{4} \alpha^2 \chi_i^A \chi_{kA} \chi^{iB} \chi_B^k \right], \quad (4.35)$$

$$K = \frac{1}{2} x^2 - t x p + t^2 H, \quad (4.36)$$

$$D = -\frac{1}{2} x p + t H, \quad (4.37)$$

$$I = \frac{i}{2} \psi_k \psi^k, \quad \bar{I} = \frac{i}{2} \bar{\psi}_k \bar{\psi}^k, \quad I_3 = \frac{i}{2} \psi_k \bar{\psi}^k, \quad (4.38)$$

$$J^{ik} = -i \left[\psi^{(i} \bar{\psi}^{k)} - \frac{1}{2} \chi^i \chi^k \right] \quad (4.39)$$

form the algebra

$$\{H, K\}_D = 2D, \quad \{H, D\}_D = H, \quad \{K, D\}_D = -K, \quad (4.40)$$

$$\{I, \bar{I}\}_D = 2I_3, \quad \{I, I_3\}_D = I, \quad \{\bar{I}, I_3\}_D = -\bar{I}, \quad (4.41)$$

$$\{J^{ij}, J^{kl}\}_D = -\epsilon^{ik} J^{jl} - \epsilon^{jl} J^{ik}. \quad (4.42)$$

Next, introducing the quantities

$$Q^{11'i} = S^i, \quad Q^{12'i} = \bar{S}^i, \quad Q^{21'i} = Q^i, \quad Q^{22'i} = \bar{Q}^i, \quad (4.43)$$

$$T^{11} = K, \quad T^{22} = H, \quad T^{12} = -D, \quad (4.44)$$

$$I^{1'1'} = I, \quad I^{2'2'} = \bar{I}, \quad I^{1'2'} = I_3, \quad (4.45)$$

we obtain that the closed superalgebra of the full set of generators takes the form

$$\{Q^{ai'i}, Q^{bk'k}\}_D = -2i \left(\epsilon^{ik} \epsilon^{i'k'} T^{ab} + \alpha \epsilon^{ab} \epsilon^{i'k'} J^{ik} - (1 + \alpha) \epsilon^{ab} \epsilon^{ik} I^{i'k'} \right), \quad (4.46)$$

$$\{T^{ab}, T^{cd}\}_D = -\epsilon^{ac} T^{bd} - \epsilon^{bd} T^{ac}, \quad (4.47)$$

$$\{I^{i'j'}, I^{k'l'}\}_D = -\epsilon^{i'k'} I^{j'l'} - \epsilon^{j'l'} I^{i'k'}, \quad (4.48)$$

$$\begin{aligned} \{T^{ab}, Q^{ci'i}\}_D &= \epsilon^{c(a} Q^{b)i'i}, \\ \{J^{ij}, Q^{ai'k}\}_D &= \epsilon^{k(i} Q^{ai'j)}, \\ \{J^{i'j'}, Q^{ak'i}\}_D &= \epsilon^{k'(i'} Q^{aj')i}. \end{aligned} \quad (4.49)$$

This is just the standard form of the superalgebra $D(2, 1; \alpha)$.

Note that the relations (4.25) - (4.49) are also valid at $\alpha = 0$, in particular, the action (4.25) is invariant under the $\alpha = 0$ version of the transformations (4.26) - (4.29). It looks somewhat paradoxical, since we started from the action (4.11) which is singular at $\alpha = 0$. The explanation is that it is just the free $(\mathbf{0}, \mathbf{4}, \mathbf{4})$ action (3.6) which is superconformal at $\alpha = 0$, and so the superconformal action of the system $(\mathbf{1}, \mathbf{4}, \mathbf{3}) \oplus (\mathbf{0}, \mathbf{4}, \mathbf{4})$ at this value of α is the sum of the actions (4.16) and (3.6), containing no interaction between the two multiplets at all. In components, after some field redefinitions, this sum of actions is reduced on shell to the $\alpha = 0$ form of (4.25). The χ_A^i part completely decouples and is reduced to the free action. The same decoupling occurs in the supercharges and the Hamiltonian. The fermions χ_A^i contribute in this case only to the $su(2)_R$ generators J^{ik} which uniformly rotate all doublet indices i, j, k .

The Casimir operators of the $su(1, 1)$, $su(2)_R$ and $su(2)_L$ subalgebras (at the classical level) defined as

$$\begin{aligned} T^2 &\equiv \frac{1}{2} T^{ab} T_{ab} = HK - D^2, \\ J^2 &\equiv \frac{1}{2} J^{ik} J_{ik}, \\ I^2 &\equiv \frac{1}{2} I^{i'k'} I_{i'k'} = I\bar{I} - (I_3)^2 \end{aligned} \quad (4.50)$$

have the following explicit form

$$T^2 = \frac{1}{8} (1 + 2\alpha) \psi_i \psi^i \bar{\psi}_k \bar{\psi}^k + \frac{1}{2} \alpha \psi_i \bar{\psi}_k \chi^{iA} \chi_A^k - \frac{1}{8} \alpha^2 \chi_i^A \chi_{kA} \chi^{iB} \chi_B^k, \quad (4.51)$$

$$J^2 = \frac{3}{8} \psi_i \psi^i \bar{\psi}_k \bar{\psi}^k + \frac{1}{2} \psi_i \bar{\psi}_k \chi^{iA} \chi_A^k - \frac{1}{8} \chi_i^A \chi_{kA} \chi^{iB} \chi_B^k, \quad (4.52)$$

$$I^2 = -\frac{3}{8} \psi_i \psi^i \bar{\psi}_k \bar{\psi}^k. \quad (4.53)$$

Using these expressions together with

$$\frac{i}{4} Q^{ai'i} Q_{ai'i} = -\frac{i}{2} (Q^i \bar{S}_i - S^i \bar{Q}_i) = \frac{1}{2} (1 + 2\alpha) \psi_i \psi^i \bar{\psi}_k \bar{\psi}^k + \alpha \psi_i \bar{\psi}_k \chi^{iA} \chi_A^k, \quad (4.54)$$

we find that the second-order (classical) Casimir operator of $D(2, 1; \alpha)$,

$$C_2 = T^2 + \alpha J^2 - (1 + \alpha) I^2 - \frac{i}{4} Q^{ai'i} Q_{ai'i}, \quad (4.55)$$

is expressed as

$$C_2 = -\frac{1}{8} \alpha (1 + \alpha) \chi_i^A \chi_{kA} \chi^{iB} \chi_B^k. \quad (4.56)$$

It is worth to point out that the additional fermionic variables χ_i^A coming from the multiplet $(\mathbf{0}, \mathbf{4}, \mathbf{4})$ make significant contributions to the $D(2, 1; \alpha)$, $su(1, 1)$ and $su(2)_R$ Casimirs (4.56), (4.51) and (4.52).

By inspecting the expressions (4.51)–(4.54), we observe that, for this particular realization of the $D(2, 1; \alpha)$ superalgebra, the following quantity identically vanishes :

$$M := T^2 - \alpha^2 J^2 - \frac{1}{3} (1 - \alpha^2) I^2 - \frac{i}{8} (1 - \alpha) Q^{ai'i} Q_{ai'i} = 0. \quad (4.57)$$

Using this identity together with the expression (4.55), we derive the constraint

$$(1 + \alpha) \left[T^2 - \alpha J^2 + \frac{1}{3} (1 - \alpha) I^2 \right] + (1 - \alpha) C_2 = 0, \quad (4.58)$$

which relates the Casimir of $D(2, 1; \alpha)$ to the Casimirs of the three mutually commuting bosonic subalgebras $su(1, 1)$, $su(2)_L$ and $su(2)_R$ in our model. Plugging the expression (4.56) for the $D(2, 1; \alpha)$ Casimir into this constraint, we find that

$$(1 + \alpha) \left[T^2 - \alpha J^2 + \frac{1}{3} (1 - \alpha) I^2 - \frac{1}{8} \alpha (1 - \alpha) \chi_i^A \chi_{kA} \chi^{iB} \chi_B^k \right] = 0. \quad (4.59)$$

Using the expressions (4.51)–(4.53), we can check that the term in the square brackets is vanishing, that is the relation

$$T^2 = \alpha J^2 - \frac{1}{3} (1 - \alpha) I^2 + \frac{1}{8} \alpha (1 - \alpha) \chi_i^A \chi_{kA} \chi^{iB} \chi_B^k \quad (4.60)$$

is valid for any α . At $\alpha = -1$ and $\alpha = 0$, the Casimir C_2 is vanishing, while at $\alpha = 1$ we have the relation $T^2 = J^2$.

Note that the Hamiltonian (4.35) can be cast in the standard form of the Hamiltonian of (super)conformal mechanics [18], [20], [19]

$$H = \frac{1}{2} p^2 + \frac{2T^2}{x^2}. \quad (4.61)$$

Using the expression (4.60), we can represent it in the convenient equivalent form

$$H = \frac{1}{2} p^2 + \alpha(1 - \alpha) \frac{\chi_i^A \chi_{kA} \chi^{iB} \chi_B^k}{8x^2} + \alpha \frac{J^2}{x^2} - (1 - \alpha) \frac{I^2}{3x^2}. \quad (4.62)$$

The last two terms involve the Casimirs of the groups $SU(2)_R$ and $SU(2)_L$.

Finally, note that the sum of the actions (4.11) and (4.20) at $\alpha = -\frac{1}{3}$ is invariant with respect to a hidden $\mathcal{N} = 8, d = 1$ supersymmetry and the exceptional $\mathcal{N} = 8$ superconformal symmetry $F(4)$ [36]. So in this special case the $D(2, 1; \alpha)$ realization given here should admit an enlargement to the appropriate realization of $F(4)$. We will not dwell on this point further.

5 The multiplet pair $(3, 4, 1) \oplus (0, 4, 4)$

5.1 The multiplet $(3, 4, 1)$

This multiplet is described by the analytic superfield $V^{++}(\zeta, u)$ subjected to the off-shell harmonic constraint [32]

$$\begin{aligned} D^{++}V^{++} &= 0 \\ \Downarrow \\ V^{++} &= v^{ik}u_i^+u_k^+ + \theta^+\varphi^iu_i^+ + \bar{\theta}^+\bar{\varphi}^iu_i^+ - i\theta^+\bar{\theta}^+ (F + 2\dot{v}^{ik}u_i^+u_k^-), \end{aligned} \quad (5.1)$$

with all the component fields being functions of t_A .

The Grassmann analyticity conditions together with the harmonic constraint (5.1), being rewritten in the central basis, imply

$$V^{++} = V^{(ik)}(t, \theta, \bar{\theta})u_i^+u_k^+, \quad D^{(i}V^{kl)} = \bar{D}^{(i}V^{kl)} = 0, \quad (5.2)$$

that is solved by

$$\begin{aligned} V^{(ik)}(t, \theta, \bar{\theta}) &= v^{ik} + \theta^{(i}\varphi^{k)} + \bar{\theta}^{(i}\bar{\varphi}^{k)} + i\theta^{(i}\bar{\theta}_l\dot{v}^{k)l} + i\theta_l\bar{\theta}^{(i}\dot{v}^{k)l} - i\theta^{(i}\bar{\theta}^{k)}F \\ &\quad - \frac{i}{2}(\theta)^2\bar{\theta}^{(i}\dot{\varphi}^{k)} - \frac{i}{2}(\bar{\theta})^2\theta^{(i}\dot{\bar{\varphi}}^{k)} + \frac{1}{4}(\theta)^2(\bar{\theta})^2\ddot{v}^{ik}, \end{aligned} \quad (5.3)$$

where $(\overline{v^{ik}}) = v_{ik}$, $(\overline{\varphi^i}) = \bar{\varphi}_i$. The $D(2, 1; \alpha)$ transformations of V^{ik} , as well as those of the component fields defined in (5.1), (5.3), were given in [32].

The superfield V^{++} have the following $D(2, 1; \alpha)$ transformation law

$$\delta V^{++} = 2\Lambda_{sc}V^{++}. \quad (5.4)$$

The full set of odd $D(2, 1; \alpha)$ transformations of the component fields reads

$$\begin{aligned} \delta v^{ik} &= -\omega^{(i}\varphi^{k)} - \bar{\omega}^{(i}\bar{\varphi}^{k)}, \\ \delta \varphi^i &= -2i\bar{\omega}_k\dot{v}^{ki} + i\bar{\omega}^iF - 4i\alpha\bar{\eta}_k v^{ki}, \quad \delta \bar{\varphi}_i = -2i\omega^k\dot{v}_{ki} - i\omega_iF - 4i\alpha\eta^k v_{ki}, \\ \delta F &= \omega^k\dot{\varphi}_k + \bar{\omega}^k\dot{\bar{\varphi}}_k - (1 - 2\alpha)(\eta^k\varphi_k + \bar{\eta}^k\bar{\varphi}_k), \end{aligned} \quad (5.5)$$

where $\omega_i = \varepsilon_i - t\eta_i$ and $\bar{\omega}^i = \bar{\varepsilon}^i - t\bar{\eta}^i$.

The general sigma-model action of V^{ik} is written as

$$S_{gen}^{(V)} = \int dt d^4\theta L(V) , \quad (5.6)$$

where $L(V)$ is an arbitrary function of V^{ik} . In order to construct the $D(2, 1; \alpha)$ invariant subclass of these actions, we use the explicit expression (5.3) to define

$$X' := \frac{1}{\sqrt{V^2}} = r' + \theta_i \varphi'^i + \bar{\varphi}'_i \bar{\theta}^i + i\theta^{(i} \bar{\theta}^{k)} A'_{ik} - \frac{i}{2} (\theta)^2 \dot{\varphi}'_i \bar{\theta}^i - \frac{i}{2} (\bar{\theta})^2 \theta_i \dot{\varphi}'^i + \frac{1}{4} (\theta)^2 (\bar{\theta})^2 \ddot{r}' , \quad (5.7)$$

where

$$r' = (v^2)^{-\frac{1}{2}} , \quad \varphi'^i = -(v^2)^{-\frac{3}{2}} v^{ik} \varphi_k , \quad \bar{\varphi}'_i = (v^2)^{-\frac{3}{2}} v_{ik} \bar{\varphi}^k , \quad (5.8)$$

$$A'_{ik} = (v^2)^{-\frac{3}{2}} [F v_{ik} - 2v_{(i}^l \dot{v}_{k)l}] + 3i(v^2)^{-\frac{5}{2}} v_{(i}^j v_{k)}^l \varphi_j \bar{\varphi}_l + \frac{i}{2} (v^2)^{-\frac{3}{2}} \varphi_{(i} \bar{\varphi}_{k)} . \quad (5.9)$$

This superfield can be checked to transform according to the superconformal transformation law (4.8) of the $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ multiplet and obey the constraints (4.1) (or, equivalently, (4.3)). It means that the superfield $1/\sqrt{V^2}$ forms some composite $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ multiplet. Therefore, the *superconformally invariant* sigma-model type actions of the multiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ are given by the following expressions [10]

$$S_{sc}^{(V)} = -\frac{1}{8(1+\alpha)} \int dt d^4\theta \left[(V^2)^{\frac{1}{2\alpha}} - (V^2)^{-\frac{1}{2}} \right] , \quad (5.10)$$

where $V^2 = V^{ik} V_{ik}$. In the limit $\alpha \rightarrow -1$ the action (5.10) is reduced to

$$S_{sc}^{(V)(\alpha=-1)} = \frac{1}{16} \int dt d^4\theta (V^2)^{-\frac{1}{2}} \log V^2 . \quad (5.11)$$

The action (5.10) has the following component form

$$\begin{aligned} S_{sc}^{(V)} &= \frac{1}{8\alpha^2} \int dt (v^2)^{\frac{1}{2\alpha}-1} \left[\dot{v}^{ik} \dot{v}_{ik} + \frac{i}{2} (\bar{\varphi}_k \dot{\varphi}^k - \dot{\bar{\varphi}}_k \varphi^k) + \frac{1}{2} F^2 \right] \\ &\quad - \frac{i}{8\alpha^2} \left(\frac{1}{\alpha} - 2 \right) \int dt (v^2)^{\frac{1}{2\alpha}-2} \left[v^{il} \dot{v}_l^k \varphi_{(i} \bar{\varphi}_{k)} + \frac{1}{2} F v^{ik} \varphi_i \bar{\varphi}_k \right] \\ &\quad - \frac{1}{96\alpha^2} \left(\frac{1}{\alpha} - 1 \right) \left(\frac{1}{\alpha} - 2 \right) \int dt (v^2)^{\frac{1}{2\alpha}-2} \varphi^i \bar{\varphi}^k \varphi_{(i} \bar{\varphi}_{k)} . \end{aligned} \quad (5.12)$$

These sigma-model type actions exist at any $\alpha \neq 0$. In what follows we will consider only the $\alpha \neq 0$ options.

Using the analyticity of V^{++} , one can construct an off-shell superpotential term for it as an integral over the analytic subspace

$$S_{sp}^{(V)} = i \frac{\gamma}{\sqrt{2}} \int du d\zeta^{--} L^{++}(V^{++}, u) . \quad (5.13)$$

The component form of this action reads

$$S_{sp}^{(V)} = \frac{\gamma}{\sqrt{2}} \int dt [F \mathcal{U}(v) + \dot{v}^{ik} \mathcal{A}_{ik}(v) - i \varphi^{(i} \bar{\varphi}^{k)} \partial_{ik} \mathcal{U}(v)] . \quad (5.14)$$

Here, the background scalar ‘half-potential’ \mathcal{U} and the magnetic one-form potential \mathcal{A}_{ik} are given by the following harmonic integrals,

$$\mathcal{U}(v) = \int du \frac{\partial L^{++}}{\partial v^{++}}, \quad \mathcal{A}_{ik}(v) = 2 \int du u_{(i}^+ u_{k)}^- \frac{\partial L^{++}}{\partial v^{++}}, \quad v^{++} = v^{ik} u_i^+ u_k^+. \quad (5.15)$$

The genuine scalar potential W appears as the result of eliminating the auxiliary field $F(t)$ in the sum of the sigma-model action (5.12) and (5.14) as

$$W(v) = -\frac{1}{4} \gamma^2 \frac{(\mathcal{U}(v))^2}{H(v)}. \quad (5.16)$$

The representation (5.15) allows one to find the most general constraints which \mathcal{U} and \mathcal{A}_{ik} should obey in order to admit an $\mathcal{N}=4$ supersymmetric extension:

$$\partial_{ik} \mathcal{A}_{lt} - \partial_{lt} \mathcal{A}_{ik} = \epsilon_{il} \partial_{kt} \mathcal{U} + \epsilon_{kt} \partial_{il} \mathcal{U}, \quad \text{and} \quad \Delta \mathcal{U} = 0. \quad (5.17)$$

The $D(2, 1; \alpha)$ invariant potential is defined by [32]

$$L_{sc}^{++} = \frac{2 \hat{V}^{++}}{\sqrt{1 + c^{--} \hat{V}^{++}} \left(1 + \sqrt{1 + c^{--} \hat{V}^{++}}\right)}, \quad (5.18)$$

where

$$\begin{aligned} \hat{V}^{++} &= V^{++} - c^{++}, \quad c^{++} = c^{ik} u_i^+ u_k^+, \quad c^2 = 2, \\ \delta_{sc} \hat{V}^{++} &= 2\Lambda_{sc}(\hat{V}^{++} + c^{++}) - 2\Lambda_{sc}^{++} c^{+-}, \quad c^{+-} = c^{ik} u_{(i}^+ u_{k)}^+. \end{aligned} \quad (5.19)$$

The analytic Lagrangian (5.18) is invariant under the transformation (5.19) up to a total harmonic derivative. This can be checked using the variation formula

$$\delta_{\hat{V}} L_{sc}^{++} = \frac{\delta \hat{V}^{++}}{(1 + c^{--} \hat{V}^{++})^{3/2}}. \quad (5.20)$$

After some algebra, one finds that

$$\delta_{sc} L_{sc}^{++} = D^{++} g, \quad g = 2\Lambda_{sc} c^{+-} \frac{2 + c^{--} \hat{V}^{++}}{(1 + c^{--} \hat{V}^{++})^{3/2}} - 2\Lambda_{sc}^{++} c^{--} \frac{1}{(1 + c^{--} \hat{V}^{++})^{1/2}}. \quad (5.21)$$

The constant triplet c^{ik} breaks spontaneously one of the two mutually commuting $SU(2)$ belonging to $D(2, 1; \alpha)$, namely, that one which rotates the indices i, k of v^{ik} and $\varphi^i, \bar{\varphi}^k$ ⁵. This triplet actually parametrizes the Dirac string, as follows from the explicit expressions for \mathcal{U} and \mathcal{A}_{ik} in the case under consideration:

$$\mathcal{U}^{conf} = \int du \frac{1}{(\sqrt{1 + c^{--} \hat{v}^{++}})^3}, \quad (5.22)$$

$$\mathcal{A}_{ik}^{conf} = 2 \int du \frac{u_{(i}^+ u_{k)}^-}{(\sqrt{1 + c^{--} \hat{v}^{++}})^3}. \quad (5.23)$$

⁵Another $SU(2)$ at any α acts only on fermions, unifying φ^i and $\bar{\varphi}^k$ into a doublet.

The harmonic integrals in (5.22) and (5.23) can be computed to give

$$\mathcal{U}^{conf} = \frac{\sqrt{2}}{\sqrt{v^{ik}v_{ik}}} = \frac{|\mathbf{c}|}{|\mathbf{v}|}, \quad (5.24)$$

$$\mathcal{A}_{ik}^{conf} = -\sqrt{2} \frac{c_i^p v_{pk} + c_k^p v_{pi}}{[(\mathbf{v} \cdot \mathbf{c}) + |\mathbf{c}||\mathbf{v}||\mathbf{v}|]}. \quad (5.25)$$

The gauge potential (5.25) is transversal

$$v^{ik} \mathcal{A}_{ik} = 0 \quad (5.26)$$

and is recognized as that of Dirac magnetic monopole, with c^{ik} parametrizing the singular Dirac string. The corresponding field strength computed by eq. (5.17) does not depend on c^{ik} and is just the Dirac monopole one

$$\partial_{ik} \mathcal{A}_{lt}^{conf} - \partial_{lt} \mathcal{A}_{ik}^{conf} = -\sqrt{2} (\epsilon_{il} v_{kt} + \epsilon_{kt} v_{il}) |\mathbf{v}|^{-3}. \quad (5.27)$$

Though the magnetic coupling $\mathcal{A}^{conf} := \mathcal{A}_{lt}^{conf} \dot{v}^{lt}$ in (5.14) explicitly includes \mathcal{A}_{lt}^{conf} , its c -dependence is reduced to the total t -derivative, as follows from the relation

$$c_{(l}^t \frac{\partial}{\partial c^m)^t} \mathcal{A}^{conf} = -\frac{\partial}{\partial t} \left(\frac{|\mathbf{c}| v_{lm} + |\mathbf{v}| c_{lm}}{(\mathbf{v} \cdot \mathbf{c}) + |\mathbf{c}||\mathbf{v}|} \right), \quad (5.28)$$

and so is vanishing under the t -integral (up to possible boundary terms).

5.2 Superconformal coupling of the multiplets (3, 4, 1) and (0, 4, 4)

Here we suggest a new way of constructing manifestly $\mathcal{N} = 4, d = 1$ conformal coupling of these two multiplets by means of generating the superconformal kinetic term of the nilpotent superfield Ψ^{+A} through the shift of the analytic superfield \hat{V}^{++} in the superpotential WZ term (5.18).

Let us define

$$W^{++} = \hat{V}^{++} + i\nu \Psi^{+A} \Psi_A^+. \quad (5.29)$$

This superfield has the same transformation properties as \hat{V}^{++} ,

$$\delta_{sc} W^{++} = 2\Lambda_{sc}(W^{++} + c^{++}) - 2\Lambda_{sc}^{++} c^{+-}, \quad (5.30)$$

and, therefore, the substitution of W^{++} for \hat{V}^{++} in (5.18) can not affect the superconformal properties of this superpotential term. Using the nilpotency property of Ψ^{+A} , the new superconformal WZ term can be written as

$$L_{sc}^{++}(W) = L_{sc}^{++}(\hat{V}) + i\nu \frac{1}{(1 + c^{--} \hat{V}^{++})^{3/2}} \Psi^{+A} \Psi_A^+. \quad (5.31)$$

Though (5.31) is guaranteed to be superconformal by construction, it is instructive to explicitly show the invariance of the second term in (5.31) (up to a total derivative),

$$S_{sc}^{(V, \Psi)} = \nu \int du d\zeta^{--} L_{sc}^{++}(V, \Psi), \quad L_{sc}^{++}(V, \Psi) := \frac{1}{(1 + c^{--} \hat{V}^{++})^{3/2}} \Psi^{+A} \Psi_A^+. \quad (5.32)$$

Under the superconformal transformations (5.19) and (3.2), the variation of the integrand in (5.32) is reduced, up to a total harmonic derivative and with making use of the constraint (3.1) and (5.1), as well as the relation $c^{++}c^{--} - (c^{+-})^2 = 1$, to

$$\delta_{sc} L_{sc}^{++}(V, \Psi) = -\frac{\Lambda_{sc}}{(1 + c^{--}\hat{V}^{++})^{3/2}} \left[1 + 3\frac{1 - (c^{+-})^2}{1 + c^{--}\hat{V}^{++}} + 15\frac{(c^{+-})^2}{(1 + c^{--}\hat{V}^{++})^2} \right] \Psi^{+A} \Psi_A^+. \quad (5.33)$$

After some algebra, denoting $Y := c^{--}\hat{V}^{++}$, the right-hand side of this variation can be represented as

$$\begin{aligned} \delta_{sc} L_{sc}^{++}(V, \Psi) &= (D^{++} f^{--}) \Psi^{+A} \Psi_A^+, \quad f^{--} = \Lambda_{sc} c^{--} c^{+-} f_1(Y) + \Lambda_{sc}^{++} (c^{--})^2 f_2(Y), \\ f_1(Y) &= -\frac{4+Y}{(1+Y)^{5/2}}, \quad f_2(Y) = \frac{1}{\sqrt{1+Y}(1+Y)}. \end{aligned} \quad (5.34)$$

Taking into account the constraint (3.1), the variation $\delta_{sc} S_{sc}^{(V, \Psi)}$ vanishes, as expected⁶.

Now we present the off-shell component form of the action in (5.32)

$$\begin{aligned} S_{sc}^{(V, \Psi)} &= \nu \int dt \left\{ 2\mathcal{U}^{conf} \left(-i\phi^{iA} \dot{\phi}_{iA} + F^A \bar{F}_A \right) - 2i\phi^{iA} \phi_A^k \dot{v}_i^l \partial_{kl} \mathcal{U}^{conf} \right. \\ &\quad + \partial_{kl} \mathcal{U}^{conf} \left[2 \left(\varphi^k \phi^{lA} \bar{F}_A - \bar{\varphi}^k \phi^{lA} F_A \right) - i\phi^{kA} \phi_A^l F \right] \\ &\quad \left. - \partial_{kl} \partial_{ij} \mathcal{U}^{conf} \varphi^{(k} \bar{\varphi}^{l)} \phi^{A(i} \phi_A^{j)} \right\}. \end{aligned} \quad (5.35)$$

Using the expression (5.24), we obtain the explicit form of the component action

$$\begin{aligned} S_{sc}^{(V, \Psi)} &= b \int dt \left\{ (v^2)^{-\frac{1}{2}} \left(-i\phi^{iA} \dot{\phi}_{iA} + F^A \bar{F}_A \right) - i(v^2)^{-\frac{3}{2}} v_i^l \dot{v}_{kl} \phi^{iA} \phi_A^k \right. \\ &\quad - (v^2)^{-\frac{3}{2}} v_{ik} \left[\left(\varphi^i \phi^{kA} \bar{F}_A - \bar{\varphi}^i \phi^{kA} F_A \right) - \frac{i}{2} \phi^{iA} \phi_A^k F \right] \\ &\quad \left. + \frac{1}{2} (v^2)^{-\frac{5}{2}} \left[(v^2) \varphi_i \bar{\varphi}_k \phi^{iA} \phi_A^k - 3 v_{kl} v_{ij} \varphi^k \bar{\varphi}^l \phi^{iA} \phi_A^j \right] \right\}, \end{aligned} \quad (5.36)$$

where

$$b := 2\sqrt{2} \nu.$$

Note that the action (5.36) coincides with the component form of the superfield action

$$S_{sc}^{(X', \Psi)} = \frac{1}{2} b \int dud\zeta^{(-2)} \mathcal{V}' \Psi^{+A} \Psi_A^+, \quad (5.37)$$

where

$$\mathcal{V}'_{WZ}(\zeta, u) = r'(t_A) - 2\theta^+ \varphi'^i(t_A) u_i^- - 2\bar{\theta}^+ \bar{\varphi}'^i(t_A) u_i^- + 3i\theta^+ \bar{\theta}^+ A'^{(ik)}(t_A) u_i^- u_k^- \quad (5.38)$$

is the analytic prepotential for the composite $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ superfield X' (cf. (4.5)).

⁶The (V, Ψ) action (5.32) was earlier derived in [12] from a different reasoning.

For what follows, it will be useful to explicitly present some intermediate on-shell form of the action (5.36) by eliminating the auxiliary fields in it. We make use of eq. (5.27) and the formulas

$$\partial_{kl}\mathcal{U}^{conf} = -\sqrt{2}v_{kl}|\mathbf{v}|^{-3}, \quad \partial_{ij}\partial_{kl}\mathcal{U}^{conf} = \frac{1}{\sqrt{2}}|\mathbf{v}|^{-5} [6v_{ij}v_{kl} - |\mathbf{v}|^2(\epsilon_{ik}\epsilon_{jl} + \epsilon_{il}\epsilon_{jk})]. \quad (5.39)$$

After substituting this into (5.36), we obtain the following expressions for the auxiliary fields and for the intermediate (V, Ψ) action

$$F^A = |\mathbf{v}|^{-2}v_{kl}\varphi^k\phi^{lA}, \quad \bar{F}^A = |\mathbf{v}|^{-2}v_{kl}\bar{\varphi}^k\phi^{lA}, \quad (5.40)$$

$$\begin{aligned} S_{sc}^{(V,\Psi)} = & b \int dt \left\{ -i|\mathbf{v}|^{-1}\phi^{iA}\dot{\phi}_{iA} + \frac{i}{2}|\mathbf{v}|^{-3} [v_{ik}F - 2v_i^l\dot{v}_{kl}] \phi^{iA}\phi_A^k \right. \\ & \left. - \frac{1}{2}|\mathbf{v}|^{-5}v_{ij}v_{kl}\varphi^{(i}\bar{\varphi}^{j)}\phi^{A(k}\phi_A^{l)} \right\}. \end{aligned} \quad (5.41)$$

After eliminating the auxiliary field F ,

$$F = -8\alpha^2\gamma(v^2)^{-\frac{1}{2\alpha}+\frac{1}{2}} + \frac{i}{2}(\frac{1}{\alpha}-2)(v^2)^{-1}v_{ik}\varphi^i\bar{\varphi}^k - 4ib\alpha^2(v^2)^{-\frac{1}{2\alpha}-\frac{1}{2}}v_{ik}\phi^{iA}\phi_A^k, \quad (5.42)$$

in the sum of (5.41) with the superconformal sigma-model action (5.12) and the superpotential WZ action (5.14) of the $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ multiplet, we obtain the ultimate on-shell component action of the coupled $(\mathbf{3}, \mathbf{4}, \mathbf{1}) \oplus (\mathbf{0}, \mathbf{4}, \mathbf{4})$ system as

$$\begin{aligned} S_{(V+\Psi)} = & \frac{1}{8\alpha^2} \int dt (v^2)^{\frac{1}{2\alpha}-1} \left[\dot{v}^{ik}\dot{v}_{ik} + \frac{i}{2} (\bar{\varphi}_k\dot{\varphi}^k - \dot{\bar{\varphi}}_k\varphi^k) \right] - ib \int dt (v^2)^{-\frac{1}{2}}\phi^{iA}\dot{\phi}_{iA} \\ & - \frac{i(1-2\alpha)}{8\alpha^3} \int dt (v^2)^{\frac{1}{2\alpha}-2}v_i^l\dot{v}_{kl}\varphi^{(i}\bar{\varphi}^{k)} - ib \int dt (v^2)^{-\frac{3}{2}}v_i^l\dot{v}_{kl}\phi^{iA}\phi_A^k \\ & + \frac{\gamma}{\sqrt{2}} \int dt \dot{v}^{ik}\mathcal{A}_{ik}(v) - 4\alpha^2\gamma^2 \int dt (v^2)^{-\frac{1}{2\alpha}} \\ & + \frac{i\gamma}{2\alpha} \int dt (v^2)^{-\frac{3}{2}}v_{ik}\varphi^i\bar{\varphi}^k - 4i\gamma b\alpha^2 \int dt (v^2)^{-\frac{1}{2\alpha}-1}v_{ik}\phi^{iA}\phi_A^k \\ & - \frac{1-2\alpha}{192\alpha^4} \int dt (v^2)^{\frac{1}{2\alpha}-2}\varphi^i\bar{\varphi}^k\varphi_{(i}\bar{\varphi}_{k)} - \frac{b}{4\alpha} \int dt (v^2)^{-\frac{5}{2}}v_{ij}v_{kl}\varphi^i\bar{\varphi}^j\phi^{Ak}\phi_A^l \\ & + \frac{\alpha^2b^2}{3} \int dt (v^2)^{-\frac{1}{2\alpha}-1}\phi^{iA}\phi_A^k\phi_i^B\phi_{kB}. \end{aligned} \quad (5.43)$$

At $b = 0$ the contribution from the multiplet $(\mathbf{0}, \mathbf{4}, \mathbf{4})$ disappears. In what follows we assume that $b \neq 0$.

Introducing the new variables

$$x = (v^2)^{\frac{1}{4\alpha}}, \quad (5.44)$$

$$\ell_{ik} = \frac{1}{2\alpha}(v^2)^{-\frac{1}{2}}v_{ik}, \quad \ell^{ik}\ell_{ik} = \frac{1}{4\alpha^2}, \quad (5.45)$$

$$\psi^i = \frac{1}{2\alpha}(v^2)^{\frac{1}{4\alpha}-1}v^{ik}\varphi_k, \quad \bar{\psi}_i = -\frac{1}{2\alpha}(v^2)^{\frac{1}{4\alpha}-1}v_{ik}\bar{\varphi}^k, \quad \chi^{iA} = \sqrt{2b}(v^2)^{-\frac{1}{4}}\phi^{iA}, \quad (5.46)$$

we recast the action (5.43) in the form

$$\begin{aligned}
S_{(V+\Psi)} = & \int dt \left\{ \frac{1}{2} \left[\dot{x}\dot{x} + x^2 \dot{\ell}^{ik} \dot{\ell}_{ik} + i \left(\bar{\psi}_k \dot{\psi}^k - \dot{\bar{\psi}}_k \psi^k \right) - i \chi^{iA} \dot{\chi}_{iA} \right] \right. \\
& + 2i \alpha \ell_i^j \dot{\ell}_{kj} \left[2\psi^{(i} \bar{\psi}^{k)} - \alpha \chi^{iA} \chi_A^k \right] + \sqrt{2} \alpha \gamma \dot{\ell}^{ik} \mathcal{B}_{ik}(\ell) \\
& - 4\alpha^2 \frac{\gamma}{x^2} \left(\gamma - 2i \ell_{ik} \psi^i \bar{\psi}^k + i \alpha \ell_{ik} \chi^{iA} \chi_A^k \right) \\
& \left. - \frac{1}{x^2} \left[\frac{1-2\alpha}{3} \psi^i \bar{\psi}^k \psi_{(i} \bar{\psi}_{k)} + 4\alpha^3 \ell_{ij} \ell_{kl} \psi^i \bar{\psi}^j \chi^{Ak} \chi_A^l - \frac{\alpha^2}{12} \chi^{iA} \chi_A^k \chi_i^B \chi_{kB} \right] \right\}. \quad (5.47)
\end{aligned}$$

It is worth pointing out that the gauge potential \mathcal{A}_{ik} defined in (5.25) is now written as

$$\mathcal{A}_{ik}(v) = x^{-2\alpha} \mathcal{B}_{ik}(\ell), \quad (5.48)$$

where

$$\mathcal{B}_{ik}(\ell) = -\frac{4\alpha c_{(i}^p \ell_{k)p}}{\sqrt{2} \alpha (\ell \cdot \mathbf{c}) + 1}. \quad (5.49)$$

The transversality condition (5.26) is rewritten as

$$\ell^{ik} \mathcal{B}_{ik} = 0. \quad (5.50)$$

Like in the previous section, one can calculate the classical $D(2,1;\alpha)$ (super)charges and their (anti)commutation relations. At $\alpha = -1$, the considered system reproduces the relevant $SU(1,1|2)$ invariant on-shell system of Ref. [27].

The on-shell supersymmetry transformations of the fields in the action (5.47) are as follows

$$\delta x = -\omega_i \psi^i + \bar{\omega}^i \bar{\psi}_i, \quad (5.51)$$

$$\delta \ell^{ik} = -2\alpha x^{-1} [\omega^{(i} \psi^{j)} + \bar{\omega}^{(i} \bar{\psi}^{j)}] \ell_j^k - 2\alpha x^{-1} [\omega^{(k} \psi^{j)} + \bar{\omega}^{(k} \bar{\psi}^{j)}] \ell_j^i, \quad (5.52)$$

$$\begin{aligned}
\delta \psi^i = & i\bar{\omega}^i \dot{x} + i\bar{\eta}^i x - 4\alpha i x \ell_m^{(i} \dot{\ell}_m^{k)} \bar{\omega}_k - 8\alpha^2 \gamma i x^{-1} \ell^{ik} \bar{\omega}_k \\
& - (1 + 2\alpha) x^{-1} \omega_k \psi^k \psi^i - 2\alpha x^{-1} \bar{\omega}_k \psi^k \bar{\psi}^i - 8\alpha^2 x^{-1} \ell^{mn} \bar{\omega}_m \psi_n \ell^{ik} \bar{\psi}_k \\
& + 4\alpha^3 x^{-1} \ell^{ik} \bar{\omega}_k \ell_{mn} \chi^{mA} \chi_A^n, \quad (5.53)
\end{aligned}$$

$$\begin{aligned}
\delta \bar{\psi}_i = & -i\omega_i \dot{x} - i\eta_i x - 4\alpha i x \ell_m^{(i} \dot{\ell}_m^{k)} \omega_k - 8\alpha^2 \gamma i x^{-1} \ell_{ik} \omega^k \\
& + (1 + 2\alpha) x^{-1} \bar{\omega}^k \bar{\psi}_k \bar{\psi}_i + 2\alpha x^{-1} \omega^k \bar{\psi}_k \psi_i + 8\alpha^2 x^{-1} \ell_{mn} \omega^m \bar{\psi}^n \ell_{ik} \psi^k \\
& + 4\alpha^3 x^{-1} \ell_{ik} \omega^k \ell_{mn} \chi^{mA} \chi_A^n, \quad (5.54)
\end{aligned}$$

$$\delta \chi^{iA} = -2\alpha x^{-1} [\omega^{(i} \psi^{j)} + \bar{\omega}^{(i} \bar{\psi}^{j)}] \chi_j^A. \quad (5.55)$$

From (5.52) and (5.55) we observe that the on-shell fermionic transformations of ℓ^{ik} and χ^{iA} have the form of field-dependent $SU(2)$ transformations like in the case of (4.29). The same property is valid for the bosonic semi-dynamical spin variables entering the $\mathcal{N} = 4$ spin multiplet as the basic ingredient of the $\mathcal{N} = 4$ SCM models of Refs. [18], [20]. This resemblance is rather interesting because in our case the angular variables ℓ^{ik} are dynamical variables.

Using the component $D(2, 1; \alpha)$ transformations (5.51), (5.52), (5.53), (5.54), (5.55), one can evaluate the corresponding Noether (super)charges

$$Q^i = p \psi^i - ix^{-1} \psi_k \left[\frac{2}{3} (1 + 2\alpha) \psi^{(i} \bar{\psi}^{k)} - \alpha \chi^{iA} \chi_A^k - 4i \alpha \left(\ell_p^i \mathcal{P}^{kp} - 2 \gamma \alpha \ell^{ik} \right) \right], \quad (5.56)$$

$$\bar{Q}_i = p \bar{\psi}_i + ix^{-1} \bar{\psi}^k \left[\frac{2}{3} (1 + 2\alpha) \psi_{(i} \bar{\psi}_{k)} - \alpha \chi_i^A \chi_{kA} - 4i \alpha \left(\ell_{ip} \mathcal{P}_k^p - 2 \gamma \alpha \ell_{ik} \right) \right], \quad (5.57)$$

$$S^i = x \psi^i - t Q^i, \quad \bar{S}_i = x \bar{\psi}_i - t \bar{Q}_i. \quad (5.58)$$

Here, $p \equiv \dot{x}$ is the canonical momentum for x , and

$$\mathcal{P}_{ik} := p_{ik} - \sqrt{2} \alpha \gamma x^{2\alpha} \mathcal{A}_{ik} = p_{ik} - \sqrt{2} \alpha \gamma \mathcal{B}_{ik},$$

with

$$p_{ik} = x^2 \dot{\ell}_{ik} + \sqrt{2} \alpha \gamma \mathcal{B}_{ik} - i \alpha \ell_i^j \left[2 \psi_{(k} \bar{\psi}_{j)} - \alpha \chi_k^A \chi_{jA} \right] - i \alpha \ell_k^j \left[2 \psi_{(i} \bar{\psi}_{j)} - \alpha \chi_i^A \chi_{jA} \right] \quad (5.59)$$

being the momenta for ℓ_{ik} . The momenta (5.59) satisfy the constraint

$$\ell^{ik} p_{ik} = 0, \quad (5.60)$$

which forms the pair of the second class constraints together with the constraint (5.45),

$$\ell^{ik} \ell_{ik} - \frac{1}{4\alpha^2} = 0. \quad (5.61)$$

Then one introduces Dirac brackets for the second class constraints (5.60), (5.61)

$$\begin{aligned} \{\ell^{ij}, \ell^{km}\}_{DB} &= 0, & \{\ell^{ij}, p_{km}\}_{DB} &= \delta_{(k}^i \delta_{m)}^j - 4\alpha^2 \ell^{ij} \ell_{km}, \\ \{p_{ij}, p_{km}\}_{DB} &= 4\alpha^2 (p_{ij} \ell_{km} - p_{km} \ell_{ij}). \end{aligned} \quad (5.62)$$

Using these Dirac brackets, one can check that the supercharges (5.56), (5.57), (5.58) form $D(2, 1; \alpha)$ algebra (4.34) with the Hamiltonian

$$\begin{aligned} H &= \frac{1}{2} p^2 + \frac{1}{x^2} \left[\frac{1}{2} \mathcal{P}^{ia} \mathcal{P}_{ia} + 4\alpha^2 \gamma^2 + 2i \alpha \left(\ell_m^i \mathcal{P}^{km} - 2\gamma \alpha \ell^{ik} \right) \left(2\psi_{(i} \bar{\psi}_{k)} - \alpha \chi_i^A \chi_{kA} \right) \right. \\ &\quad \left. + \frac{1}{4} (1 + 2\alpha) \psi_i \psi^i \bar{\psi}_k \bar{\psi}^k + \alpha \psi_i \bar{\psi}_k \chi^{iA} \chi_A^k - \frac{1}{4} \alpha^2 \chi_i^A \chi_{kA} \chi^{iB} \chi_B^k \right] \end{aligned} \quad (5.63)$$

and $SU(2)$ generators

$$J^{ik} = -i \left[\psi^{(i} \bar{\psi}^{k)} - \frac{1}{2} \chi^{iA} \chi_A^k - 2i \left(\ell_m^i \mathcal{P}^{km} - 2\gamma \alpha \ell^{ik} \right) \right]. \quad (5.64)$$

The remaining even generators K , D and $I^{i'k'}$ are given by the same expressions as in (4.36), (4.37) and (4.38), (4.45).

Note the useful relations

$$\ell_m^i p^{km} = \ell_m^{(i} p^{k)m}, \quad \ell_m^i p^{km} \ell_{ij} p_k^j = \frac{1}{8\alpha^2} p^{km} p_{km}, \quad (5.65)$$

$$\left(\ell_m^i \mathcal{P}^{km} - 2\gamma\alpha \ell^{ik}\right) \left(\ell_{ij} \mathcal{P}_k^j - 2\gamma\alpha \ell_{ik}\right) = \frac{1}{8\alpha^2} \left(\frac{1}{2} \mathcal{P}^{ia} \mathcal{P}_{ia} + 4\alpha^2 \gamma^2\right), \quad (5.66)$$

$$\ell_m^i \mathcal{P}^{km} - 2\gamma\alpha \ell^{ik} = \ell_m^i p^{km} - \gamma \frac{2\alpha |\mathbf{c}| \ell^{ik} + c^{ik}}{2\alpha(\ell \cdot \mathbf{c}) + |\mathbf{c}|}, \quad (5.67)$$

which follow from the definitions of ℓ^{ik} , \mathcal{P}_{ik} and p_{ik} . Using these relations, we can represent the Casimir operators (4.50) of the $su(1,1)$, $su(2)_R$ and $su(2)_L$ subalgebras as

$$\begin{aligned} T^2 &= \frac{1}{8} (1 + 2\alpha) \psi_i \psi^i \bar{\psi}_k \bar{\psi}^k + \frac{1}{2} \alpha \psi_i \bar{\psi}_k \left[\chi^{iA} \chi_A^k + 4i \left(\ell_m^i \mathcal{P}^{km} - 2\gamma\alpha \ell^{ik} \right) \right] \\ &\quad - \frac{\alpha^2}{8} \left[\chi_i^A \chi_{kA} + 4i \left(\ell_{im} \mathcal{P}_k^m - 2\gamma\alpha \ell_{ik} \right) \right] \left[\chi^{iB} \chi_B^k + 4i \left(\ell_p^i \mathcal{P}^{kp} - 2\gamma\alpha \ell^{ik} \right) \right], \end{aligned} \quad (5.68)$$

$$\begin{aligned} J^2 &= \frac{3}{8} \psi_i \psi^i \bar{\psi}_k \bar{\psi}^k + \frac{1}{2} \psi_i \bar{\psi}_k \left[\chi^{iA} \chi_A^k + 4i \left(\ell_m^i \mathcal{P}^{km} - 2\gamma\alpha \ell^{ik} \right) \right] \\ &\quad - \frac{1}{8} \left[\chi_i^A \chi_{kA} + 4i \left(\ell_{im} \mathcal{P}_k^m - 2\gamma\alpha \ell_{ik} \right) \right] \left[\chi^{iB} \chi_B^k + 4i \left(\ell_p^i \mathcal{P}^{kp} - 2\gamma\alpha \ell^{ik} \right) \right], \end{aligned} \quad (5.69)$$

$$I^2 = -\frac{3}{8} \psi_i \psi^i \bar{\psi}_k \bar{\psi}^k. \quad (5.70)$$

Exploiting these expressions and the relation

$$\frac{i}{4} Q^{ai'i} Q_{ai'i} = \frac{1}{2} (1 + 2\alpha) \psi_i \psi^i \bar{\psi}_k \bar{\psi}^k + \alpha \psi_i \bar{\psi}_k \left[\chi^{iA} \chi_A^k + 4i \left(\ell_m^i \mathcal{P}^{km} - 2\gamma\alpha \ell^{ik} \right) \right], \quad (5.71)$$

we obtain the second-order (classical) Casimir operator of $D(2,1;\alpha)$ in the form

$$C_2 = -\frac{1}{8} \alpha(1 + \alpha) \left[\chi_i^A \chi_{kA} + 4i \left(\ell_{im} \mathcal{P}_k^m - 2\gamma\alpha \ell_{ik} \right) \right] \left[\chi^{iB} \chi_B^k + 4i \left(\ell_p^i \mathcal{P}^{kp} - 2\gamma\alpha \ell^{ik} \right) \right]. \quad (5.72)$$

Note that, as in the case of the pair $(\mathbf{1}, \mathbf{4}, \mathbf{3}) \oplus (\mathbf{0}, \mathbf{4}, \mathbf{4})$, the additional fermionic variables χ_i^A make significant contributions to the Casimirs (5.68), (5.69) and (5.72).

Like in the previous case, looking at the expressions (5.68)–(5.71), we observe that the following quantity M vanishes identically for this particular realization of the $D(2,1;\alpha)$ superalgebra:

$$M \equiv T^2 - \alpha^2 J^2 - \frac{1}{3} (1 - \alpha^2) I^2 - \frac{i}{8} (1 - \alpha) Q^{ai'i} Q_{ai'i} = 0. \quad (5.73)$$

Taking into account the expression (5.71), this identity implies the constraint

$$(1 + \alpha) \left[T^2 - \alpha J^2 + \frac{1}{3} (1 - \alpha) I^2 \right] + (1 - \alpha) C_2 = 0, \quad (5.74)$$

which relates the Casimir of $D(2,1;\alpha)$ to the Casimirs of the three mutually commuting bosonic subalgebras $su(1,1)$, $su(2)_L$ and $su(2)_R$ in our model. Plugging the expression (5.72) for the $D(2,1;\alpha)$ Casimir into this constraint, we find (cf. (4.59)):

$$\begin{aligned} &(1 + \alpha) \left\{ T^2 - \alpha J^2 + \frac{1}{3} (1 - \alpha) I^2 - \frac{1}{8} \alpha(1 - \alpha) \left[\chi_i^A \chi_{kA} + 4i \left(\ell_{im} \mathcal{P}_k^m - 2\gamma\alpha \ell_{ik} \right) \right] \right. \\ &\quad \times \left. \left[\chi^{iB} \chi_B^k + 4i \left(\ell_p^i \mathcal{P}^{kp} - 2\gamma\alpha \ell^{ik} \right) \right] \right\} = 0. \end{aligned}$$

Using the expressions (5.68)–(5.70), we can check that the expression within the curled brackets is vanishing on its own, whence

$$\begin{aligned} T^2 &= \alpha J^2 - \frac{1}{3} (1 - \alpha) I^2 + \frac{1}{8} \alpha(1 - \alpha) \left[\chi_i^A \chi_{kA} + 4i \left(\ell_{im} \mathcal{P}_k^m - 2\gamma\alpha \ell_{ik} \right) \right] \times \\ &\quad \times \left[\chi^{iB} \chi_B^k + 4i \left(\ell_p^i \mathcal{P}^{kp} - 2\gamma\alpha \ell^{ik} \right) \right]. \end{aligned}$$

Note that the Hamiltonian (5.63) has the standard form of the Hamiltonian of (super)conformal mechanics

$$H = \frac{1}{2} p^2 + \frac{2T^2}{x^2}. \quad (5.75)$$

Using the relation (5.75), we can rewrite the Hamiltonian in the form

$$\begin{aligned} H = & \frac{1}{2} p^2 + \alpha \frac{J^2}{x^2} - (1 - \alpha) \frac{I^2}{3x^2} + \frac{\alpha(1 - \alpha)}{8x^2} \left[\chi_i^A \chi_{kA} + 4i \left(\ell_{im} \mathcal{P}_k^m - 2\gamma\alpha \ell_{ik} \right) \right] \times \\ & \times \left[\chi^{iB} \chi_B^k + 4i \left(\ell_p^i \mathcal{P}^{kp} - 2\gamma\alpha \ell^{ik} \right) \right]. \end{aligned} \quad (5.76)$$

This representation is analogous to (4.62).

6 The multiplet pair $(4, 4, 0) \oplus (0, 4, 4)$

6.1 The multiplet $(4, 4, 0)$

This multiplet is described by the analytic superfield $q_a^+(\zeta, u)$, $\widetilde{q_a^+} = \epsilon^{ab} q_b^+ := q^{+a}$, $\widetilde{q^{+a}} = -q_a^+$, subjected to the off-shell harmonic constraint [32]

$$\begin{aligned} D^{++} q^{+a} &= 0 \\ \Downarrow \\ q^{+a} &= f^{ia} u_i^+ + \theta^+ \varphi^a + \bar{\theta}^+ \bar{\varphi}^a - 2i\theta^+ \bar{\theta}^+ \dot{f}^{ia} u_i^-. \end{aligned} \quad (6.1)$$

Here $a = 1, 2$ is the doublet index of some extra ‘‘Pauli-Gürsey’’ $SU(2)$ which commutes with both the Poincaré and conformal $\mathcal{N}=4, d=1$ supersymmetries. The harmonic constraint (6.1) together with the analyticity conditions amount in the central basis to [11]

$$q^{+a} = q^{ia}(t, \theta, \bar{\theta}) u_i^+, \quad D^{(i} q^{k)a} = \bar{D}^{(i} q^{k)a} = 0, \quad (6.2)$$

where

$$q^{ia}(t, \theta_i, \bar{\theta}^i) = f^{ia} + \theta^i \varphi^a + \bar{\theta}^i \bar{\varphi}^a - 2i\theta^i \bar{\theta}^k \dot{f}_k^a - \frac{i}{2}(\theta)^2 \bar{\theta}^i \dot{\varphi}^a - \frac{i}{2}(\bar{\theta})^2 \theta^i \dot{\bar{\varphi}}^a + \frac{1}{4}(\theta)^2 (\bar{\theta})^2 \ddot{f}^{ia}. \quad (6.3)$$

Under the superconformal symmetry the analytic superfield q^{+a} transforms as

$$\delta_{sc} q^{+a} = \Lambda_{sc} q^{+a}. \quad (6.4)$$

The full set of the supersymmetry transformations acts on the component fields as

$$\begin{aligned} \delta f^{ia} &= -\omega^i \varphi^a - \bar{\omega}^i \bar{\varphi}^a, \\ \delta \varphi^a &= -2i\bar{\omega}_k \dot{f}^{ka} - 2i\alpha \bar{\eta}_k f^{ka}, \quad \delta \bar{\varphi}_a = -2i\omega^k \dot{f}_{ka} - 2i\alpha \eta^k f_{ka}, \end{aligned} \quad (6.5)$$

where $\omega_i = \varepsilon_i - t \eta_i$ and $\bar{\omega}^i = \bar{\varepsilon}^i - t \bar{\eta}^i$.

The general sigma-model type q^+ -action reads

$$S_{gen}^{(q)} = \int dt d^4\theta L(q^{ia}), \quad (6.6)$$

with $L(q)$ being an arbitrary function of q^{ia} .

Using the explicit component expansion (6.3) and defining $(q^2)^{-1} \equiv (q^{ia}q_{ia})^{-1}$, we can construct the composite superfield

$$X'' \equiv \frac{1}{q^2} = r'' + \theta_i \varphi'''^i + \bar{\varphi}_i'' \bar{\theta}^i + i\theta^{(i} \bar{\theta}^{k)} A''_{ik} - \frac{i}{2}(\theta)^2 \dot{\varphi}_i'' \bar{\theta}^i - \frac{i}{2}(\bar{\theta})^2 \theta_i \dot{\varphi}'''^i + \frac{1}{4}(\theta)^2 (\bar{\theta})^2 \ddot{r}'' , \quad (6.7)$$

where

$$r'' = (f^2)^{-1}, \quad \varphi'''^i = -2(f^2)^{-2} f^{ia} \varphi_a, \quad \bar{\varphi}_i'' = 2(f^2)^{-2} f_{ia} \bar{\varphi}^a, \quad (6.8)$$

$$A''_{ik} = -4(f^2)^{-2} f_{(i} \dot{f}_{k)a} + 8i(f^2)^{-3} f_{(i}^a f_{k)}^b \varphi_a \bar{\varphi}_b \quad (6.9)$$

and $f^2 \equiv f^{ia} f_{ia}$. One can check that X'' transforms under $D(2, 1; \alpha)$ as the standard $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ superfield X and obeys the same constraints. Hence, the superfield $1/q^2$ represents a composite $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ multiplet. Therefore, the *superconformally invariant* sigma-model type actions of the multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ can be constructed just on the pattern of the X superconformal action (4.11):

$$S_{sc}^{(q)} = -\frac{1}{8(1+\alpha)} \int dt d^4\theta \left[(q^2)^{\frac{1}{\alpha}} - (q^2)^{-1} \right], \quad (6.10)$$

where $q^2 = q^{ia} q_{ia}$. In the limit $\alpha \rightarrow -1$ the action (6.10) goes over to

$$S_{sc}^{(q)(\alpha=-1)} = \frac{1}{8} \int dt d^4\theta (q^2)^{-1} \log q^2. \quad (6.11)$$

The action (6.10) has the following component formulation

$$\begin{aligned} S_{sc}^{(q)} &= \frac{1}{2\alpha^2} \int dt (f^2)^{\frac{1}{\alpha}-1} \left[\dot{f}^{ia} \dot{f}_{ia} + \frac{i}{2} (\bar{\varphi}_a \dot{\varphi}^a - \dot{\bar{\varphi}}_a \varphi^a) \right] \\ &\quad - \frac{i}{\alpha^2} \left(\frac{1}{\alpha} - 1 \right) \int dt (f^2)^{\frac{1}{\alpha}-2} f^{ia} \dot{f}_i^b \varphi_{(a} \bar{\varphi}_{b)} \\ &\quad - \frac{1}{6\alpha^3} \left(\frac{1}{\alpha} - 1 \right) \int dt (f^2)^{\frac{1}{\alpha}-2} \varphi^a \bar{\varphi}^b \varphi_{(a} \bar{\varphi}_{b)}. \end{aligned} \quad (6.12)$$

The component bosonic actions is meaningful only under the assumption that the ‘vacuum value’ of the radial part of q^{ia} is non-vanishing, i.e. $\langle q^2 \rangle \neq 0$. It is worth noting that the extra Pauli-Gürsey $SU(2)$ group acting on the index a of q^{ia} is respected by the superconformal actions (6.10), (6.12). Thus these actions are manifestly $O(4)$ invariant.

For the $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplet one can also define the superpotential WZ term

$$S_{sp}^{(q)} = -\frac{i}{2} \int du d\zeta^{(-2)} \mathcal{L}^{++}(q^+, u), \quad (6.13)$$

which in components yields the coupling $\sim \dot{f}^{ia} \mathcal{A}_{ia}$, as in the case of the $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ multiplet. However, as opposed to the latter case, the superconformal subclass of (6.13) is vanishing because the corresponding component Lagrangian is reduced to a total t -derivative. This conclusion is based on the following reasoning. Define

$$\mathcal{V}^{++} = q^{+a} a_{ab} q^{+b}, \quad a^{ab} a_{ab} = 2, \quad (6.14)$$

where $a_{ab} = a_{ba}$ is a constant triplet which breaks the extra Pauli-Gürsey $SU(2)$ (realized on the indices a, b) down to its some $U(1)$ subgroup. The composite analytic superfield \mathcal{V}^{++}

has the same transformation rule under $D(2, 1; \alpha)$ as V^{++} , $\delta\mathcal{V}^{++} = 2\Lambda_{sc}\mathcal{V}^{++}$, so one can apply the same method of constructing a superconformal WZ term for it. After substituting the component fields of \mathcal{V}^{++} ,

$$\mathcal{V}_0^{ik} = f^{ia}a_{ab}f^{kb}, \quad \mathcal{F} = 2(\dot{f}_i^a a_{ab}f^{ib} + i\varphi^a a_{ab}\bar{\varphi}^b), \quad \chi^i = 2\varphi^a a_{ab}f^{ib}, \quad \bar{\chi}^i = 2\bar{\varphi}^a a_{ab}f^{ib}, \quad (6.15)$$

into (5.14), the fermionic contributions from the first and third terms in (5.14) cancel each other, while the bosonic rest is reduced to

$$S_{sp(sc)}^{(q)} \implies \int dt \dot{f}^{ia} \mathcal{A}_{ia}^{conf}(f), \quad (6.16)$$

where

$$\mathcal{A}_{ia}^{conf}(f) = \frac{4}{[f^2 + (f \cdot a \cdot f \cdot c)]} (f_a^k c_{ki} - f_i^b a_{ab}) \quad (6.17)$$

and

$$f \cdot a \cdot f \cdot c = f^{ia} f^{kb} c_{ik} a_{ab}.$$

Calculating the curl of the vector potential (6.17), we find

$$\partial_{kb} \mathcal{A}_{ia}^{conf} - \partial_{ia} \mathcal{A}_{kb}^{conf} = 0, \quad (6.18)$$

i.e. \mathcal{A}_{ia}^{conf} is a pure gauge and the Lagrangian in (6.16) is a total time derivative.

6.2 Superconformal couplings of the multiplets $(4, 4, 0)$ and $(0, 4, 4)$

Similarly to (4.5), we can construct the “prepotential” for the superfield $1/q^2$:

$$\mathcal{V}_{WZ}''(\zeta, u) = r''(t_A) - 2\theta^+ \varphi''^i(t_A) u_i^- - 2\bar{\theta}^+ \bar{\varphi}''^i(t_A) u_i^- + 3i\theta^+ \bar{\theta}^+ A''^{(ik)}(t_A) u_i^- u_k^-, \quad (6.19)$$

where r'' , φ''^i , $\bar{\varphi}''^i$ and $A''^{(ik)}$ are defined in (6.8), (6.9).

Then the superconformal coupling is given by the superfield action

$$S_{sc}^{(X'', \Psi)} = \frac{1}{2} b \int d\zeta d\bar{\zeta} \mathcal{V}'' \Psi^{+A} \Psi_A^+. \quad (6.20)$$

It is easy to specialize the component superconformal action (4.21) to this case:

$$\begin{aligned} S_{sc}^{(q, \Psi)} = b \int dt \Bigg\{ & (f^2)^{-1} \left(-i\phi^{iA} \dot{\phi}_{iA} + F^A \bar{F}_A \right) - 2i(f^2)^{-2} f_i^a \dot{f}_{ka} \phi^{iA} \phi_A^k \\ & - 2(f^2)^{-2} f_{ka} \left(\varphi^a \phi^{kA} \bar{F}_A - \bar{\varphi}^a \phi^{kA} F_A \right) \\ & - 4(f^2)^{-3} f_{ia} f_{kb} \varphi^a \bar{\varphi}^b \phi^{iA} \phi_A^k \Bigg\}. \end{aligned} \quad (6.21)$$

Elimination of the auxiliary fields F^A, \bar{F}_A by their equations of motion,

$$F^A = 2(f^2)^{-1} f_{ka} \varphi^a \phi^{kA}, \quad \bar{F}_A = 2(f^2)^{-1} f^{ka} \bar{\varphi}_a \phi_{kA}, \quad (6.22)$$

nullifies the total four-fermionic term in (6.21), resulting in the very simple on-shell action

$$S_{sc}^{(q,\Psi)} = -ib \int dt (f^2)^{-1} \left[\phi^{iA} \dot{\phi}_{iA} + 2(f^2)^{-1} f_i^a \dot{f}_{ka} \phi^{iA} \phi_A^k \right]. \quad (6.23)$$

The same superconformal coupling can be constructed just through the substitution $\hat{\mathcal{V}}^{++} \rightarrow \mathcal{V}^{++}$ in (5.31), where the composite analytic superfield \mathcal{V}^{++} was defined in (6.14). While $L_{sc}^{++}(\mathcal{V})$ is reduced to the total derivative and so does not contribute, the rest of (5.31) is a non-trivial $D(2, 1; \alpha)$ invariant off-shell coupling of the multiplets $(\mathbf{0}, \mathbf{4}, \mathbf{4})$ and $(\mathbf{4}, \mathbf{4}, \mathbf{0})$:

$$S_{sc}^{(q,\Psi)} = b \int dud\zeta^{--} L_{sc}^{++}(q, \Psi), \quad L_{sc}^{++}(q, \Psi) := \frac{1}{(1 + c^{--} \hat{\mathcal{V}}^{++})^{3/2}} \Psi^{+A} \Psi_A^+. \quad (6.24)$$

In components, it yields the same actions (6.21) and (6.23).

The total on-shell superconformal action is the sum of the component superconformal $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ action (6.12) and the action (6.23):

$$\begin{aligned} S_{(q+\Psi)} &= \frac{1}{2\alpha^2} \int dt (f^2)^{\frac{1}{\alpha}-1} \left[\dot{f}^{ia} \dot{f}_{ia} + \frac{i}{2} (\bar{\varphi}_a \dot{\varphi}^a - \dot{\bar{\varphi}}_a \varphi^a) \right] - ib \int dt (f^2)^{-1} \phi^{iA} \dot{\phi}_{iA} \\ &\quad - \frac{i}{\alpha^2} \left(\frac{1}{\alpha} - 1 \right) \int dt (f^2)^{\frac{1}{\alpha}-2} f^{ia} \dot{f}_i^b \varphi_{(a} \bar{\varphi}_{b)} - 2ib \int dt (f^2)^{-2} f_i^a \dot{f}_{ka} \phi^{iA} \phi_A^k \\ &\quad - \frac{1}{6\alpha^3} \left(\frac{1}{\alpha} - 1 \right) \int dt (f^2)^{\frac{1}{\alpha}-2} \varphi^a \bar{\varphi}^b \varphi_{(a} \bar{\varphi}_{b)}. \end{aligned} \quad (6.25)$$

Thus in this case fermionic fields from different multiplets interact only with bosonic fields and the four-fermionic term is composed only out of the fermionic fields of the $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplet.

Introducing the new variables

$$x = (f^2)^{\frac{1}{2\alpha}}, \quad (6.26)$$

$$L_{ia} = \frac{1}{\alpha} (f^2)^{-\frac{1}{2}} f_{ia}, \quad L^{ia} L_{ia} = \frac{1}{\alpha^2}, \quad (6.27)$$

$$\psi^i = \frac{1}{\alpha} (f^2)^{\frac{1}{2\alpha}-1} f^{ia} \varphi_a, \quad \bar{\psi}_i = -\frac{1}{\alpha} (f^2)^{\frac{1}{2\alpha}-1} f_{ia} \bar{\varphi}^a, \quad \chi^{iA} = \sqrt{2b} (f^2)^{-\frac{1}{2}} \phi^{iA}, \quad (6.28)$$

at $\alpha \neq 0$ we can rewrite the action (6.23) in the form

$$\begin{aligned} S_{(q+\Psi)} &= \int dt \left\{ \frac{1}{2} \left[\dot{x} \dot{x} + x^2 L^{ia} \dot{L}_{ia} + i \left(\bar{\psi}_k \dot{\psi}^k - \dot{\bar{\psi}}_k \psi^k \right) - i \chi^{iA} \dot{\chi}_{iA} \right] \right. \\ &\quad \left. + i \alpha L_i^a \dot{L}_{ka} \left[2\psi^{(i} \bar{\psi}^{k)} - \alpha \chi^{iA} \chi_A^k \right] - \frac{2(1-\alpha)}{3} x^{-2} \psi^i \bar{\psi}^k \psi_{(i} \bar{\psi}_{k)} \right\}. \end{aligned} \quad (6.29)$$

The on-shell supersymmetry transformations leaving the action (6.29) invariant are

$$\delta x = -\omega_i \psi^i + \bar{\omega}^i \bar{\psi}_i, \quad (6.30)$$

$$\delta L^{ia} = -2\alpha x^{-1} [\omega^{(i} \psi^{j)} + \bar{\omega}^{(i} \bar{\psi}^{j)}] L_j^a, \quad (6.31)$$

$$\begin{aligned} \delta \psi^i &= i\bar{\omega}^i \dot{x} + i\bar{\eta}^i x - 2\alpha i x L^{ia} \dot{L}_a^k \bar{\omega}_k \\ &\quad - (1 + 2\alpha) x^{-1} \omega_k \psi^k \psi^i - 2\alpha x^{-1} \bar{\omega}_k \psi^k \bar{\psi}^i + x^{-1} \bar{\omega}_k \psi^i \bar{\psi}^k, \end{aligned} \quad (6.32)$$

$$\begin{aligned} \delta \bar{\psi}_i &= -i\omega_i \dot{x} - i\eta_i x - 2\alpha i x L_i^a \dot{L}_{ka} \omega^k \\ &\quad + (1 + 2\alpha) x^{-1} \bar{\omega}^k \bar{\psi}_k \bar{\psi}_i + 2\alpha x^{-1} \omega^k \bar{\psi}_k \psi_i - x^{-1} \omega^k \bar{\psi}_i \psi_k, \end{aligned} \quad (6.33)$$

$$\delta \chi^{iA} = -2\alpha x^{-1} [\omega^{(i} \psi^{j)} + \bar{\omega}^{(i} \bar{\psi}^{j)}] \chi_j^A. \quad (6.34)$$

Once again, from (6.31) and (6.34) we notice that the fermionic transformations of the fields L^{ia} and χ^{iA} look as the field-dependent $SU(2)$ transformations of the doublet indices of these fields.

Proceeding from the component $D(2, 1; \alpha)$ transformations (6.30) - (6.34), one can construct the corresponding Noether (super)charges

$$Q^i = p \psi^i - ix^{-1} \psi_k \left[\frac{2}{3} (1 + 2\alpha) \psi^{(i} \bar{\psi}^{k)} - \alpha \chi^{iA} \chi_A^k - 2i \alpha L^{(i}{}_a p^{k)a} \right], \quad (6.35)$$

$$\bar{Q}_i = p \bar{\psi}_i + ix^{-1} \bar{\psi}^k \left[\frac{2}{3} (1 + 2\alpha) \psi_{(i} \bar{\psi}_{k)} - \alpha \chi_i^A \chi_{kA} + 2i \alpha L_{(i}{}^a p_{k)a} \right], \quad (6.36)$$

$$S^i = x \psi^i - t Q^i, \quad \bar{S}_i = x \bar{\psi}_i - t \bar{Q}_i. \quad (6.37)$$

Here $p \equiv \dot{x}$ is the canonical momentum for x and

$$p_{ia} = x^2 \dot{L}_{ia} - i \alpha L^k{}_a \left[2 \psi_{(i} \bar{\psi}_{k)} - \alpha \chi_i^A \chi_{kA} \right] \quad (6.38)$$

are the momenta for L^{ia} . The momenta (6.38) satisfy the constraint

$$L^{ia} p_{ia} = 0, \quad (6.39)$$

which forms the pair of second class constraints together with the constraint (6.27),

$$L^{ia} L_{ia} - \frac{1}{\alpha^2} = 0. \quad (6.40)$$

Like in subsection 5.2, the presence of second class constraints (6.39), (6.40) implies that in the present case one should use the Dirac brackets

$$\begin{aligned} \{L^{ia}, L^{kb}\}_{DB} &= 0, \quad \{L^{ia}, p_{kb}\}_{DB} = \delta_k^i \delta_b^a - \alpha^2 L^{ia} L_{kb}, \\ \{p_{ia}, p_{kb}\}_{DB} &= \alpha^2 (p_{ia} L_{kb} - p_{kb} L_{ia}). \end{aligned} \quad (6.41)$$

Making use of them, one can directly check that the supercharges (6.36), (6.35), (6.37) form $D(2, 1; \alpha)$ algebra (4.34) with the Hamiltonian

$$\begin{aligned} H &= \frac{1}{2} p^2 + \frac{1}{x^2} \left[\frac{1}{2} p^{ia} p_{ia} + i \alpha L^i{}_a p^{ka} \left(2 \psi_{(i} \bar{\psi}_{k)} - \alpha \chi_i^A \chi_{kA} \right) \right. \\ &\quad \left. + \frac{1}{4} (1 + 2\alpha) \psi_i \psi^i \bar{\psi}_k \bar{\psi}^k + \alpha \psi_i \bar{\psi}_k \chi^{iA} \chi_A^k - \frac{1}{4} \alpha^2 \chi_i^A \chi_{kA} \chi^{iB} \chi_B^k \right] \end{aligned} \quad (6.42)$$

and the $SU(2)$ generators

$$J^{ik} = -i \left[\psi^{(i} \bar{\psi}^{k)} - \frac{1}{2} \chi^{iA} \chi_A^k - i L^i{}_a p^{ka} \right]. \quad (6.43)$$

The remaining even generators K , D and $I^{i'k'}$ are defined by the same expressions as in (4.36), (4.37) and (4.38), (4.45).

Note the relations

$$L^i{}_a p^{ka} = L^{(i}{}_a p^{k)a}, \quad L^i{}_a p^{ka} L_{ib} p_k{}^b = \frac{1}{2\alpha^2} p^{ka} p_{ka}. \quad (6.44)$$

With the help of them, one can find the explicit form of the Casimir operators (4.50) of the $su(1, 1)$, $su(2)_R$ and $su(2)_L$ algebras for the case under consideration

$$T^2 = \frac{1}{8} (1 + 2\alpha) \psi_i \psi^i \bar{\psi}_k \bar{\psi}^k + \frac{1}{2} \alpha \psi_i \bar{\psi}_k (\chi^{iA} \chi_A^k + 2i L^i_a p^{ka}) - \frac{1}{8} \alpha^2 (\chi_i^A \chi_{kA} + 2i L_{ia} p_k^a) (\chi^{iB} \chi_B^k + 2i L^i_b p^{kb}), \quad (6.45)$$

$$J^2 = \frac{3}{8} \psi_i \psi^i \bar{\psi}_k \bar{\psi}^k + \frac{1}{2} \psi_i \bar{\psi}_k (\chi^{iA} \chi_A^k + 2i L^i_a p^{ka}) - \frac{1}{8} (\chi_i^A \chi_{kA} + 2i L_{ia} p_k^a) (\chi^{iB} \chi_B^k + 2i L^i_b p^{kb}), \quad (6.46)$$

$$I^2 = -\frac{3}{8} \psi_i \psi^i \bar{\psi}_k \bar{\psi}^k. \quad (6.47)$$

Using these expressions and the expression

$$\frac{i}{4} Q^{ai'i} Q_{ai'i} = \frac{1}{2} (1 + 2\alpha) \psi_i \psi^i \bar{\psi}_k \bar{\psi}^k + \alpha \psi_i \bar{\psi}_k (\chi^{iA} \chi_A^k + 2i L^i_a p^{ka}), \quad (6.48)$$

we find that the second-order (classical) Casimir operator of $D(2, 1; \alpha)$ takes the form

$$C_2 = -\frac{1}{8} \alpha (1 + \alpha) (\chi_i^A \chi_{kA} + 2i L_{ia} p_k^a) (\chi^{iB} \chi_B^k + 2i L^i_b p^{kb}). \quad (6.49)$$

Like in the previous cases, the additional fermionic variables χ_i^A make significant contributions to the $D(2, 1; \alpha)$, $su(1, 1)$ and $su(2)_R$ Casimirs (6.45), (6.46), (6.49).

By the same tokens as in the previous cases, we obtain the various relations between the Casimirs pertinent to the concrete realization of $D(2, 1; \alpha)$ we have constructed in this section. These relations are

$$M \equiv T^2 - \alpha^2 J^2 - \frac{1}{3} (1 - \alpha^2) I^2 - \frac{i}{8} (1 - \alpha) Q^{ai'i} Q_{ai'i} = 0, \quad (6.50)$$

$$(1 + \alpha) \left[T^2 - \alpha J^2 + \frac{1}{3} (1 - \alpha) I^2 \right] + (1 - \alpha) C_2 = 0, \quad (6.51)$$

$$T^2 = \alpha J^2 - \frac{1}{3} (1 - \alpha) I^2 + \frac{1}{8} \alpha (1 - \alpha) (\chi_i^A \chi_{kA} + 2i L_{ia} p_k^a) (\chi^{iB} \chi_B^k + 2i L^i_b p^{kb}). \quad (6.52)$$

All these relations are valid for any α , including $\alpha = -1$.

The Hamiltonian (6.42) can be cast in the standard form of the Hamiltonian of (super)conformal mechanics

$$H = \frac{1}{2} p^2 + \frac{2T^2}{x^2}. \quad (6.53)$$

Using the relation (6.52), we can rewrite it in the equivalent form as

$$H = \frac{1}{2} p^2 + \alpha \frac{J^2}{x^2} - (1 - \alpha) \frac{I^2}{3x^2} + \alpha (1 - \alpha) \frac{(\chi_i^A \chi_{kA} + 2i L_{ia} p_k^a) (\chi^{iB} \chi_B^k + 2i L^i_b p^{kb})}{8x^2}. \quad (6.54)$$

7 Superconformal coupling of the multiplets $(1, 4, 3)$ and $(4, 4, 0)$ mediated by the multiplet $(0, 4, 4)$

Here we illustrate the efficiency of the off-shell superfield approach for constructing new superconformal systems which involve dynamical supermultiplets of different types. Here we construct a system in which the superconformal interaction between the multiplets $(1, 4, 3)$ and $(4, 4, 0)$ arises as a result of coupling of these both multiplets to the single $(0, 4, 4)$ multiplet.

To this end, we will consider a sum of the superfield actions (4.11), (4.20) which describe the superconformal coupling of the $(1, 4, 3)$ and $(0, 4, 4)$ multiplets and the actions (6.10), (6.20) which describe an analogous coupling of the $(4, 4, 0)$ and $(0, 4, 4)$ multiplets. After elimination of the auxiliary fields in this sum, we will obtain a new superconformal coupling of the $(1, 4, 3)$ and $(4, 4, 0)$ multiplets mediated by the multiplet $(0, 4, 4)$.

The corresponding off-shell component action is the sum of the component actions (4.13), (4.21), (6.12), (6.21). To distinguish between contributions of different actions, we substituted the coupling constant as $b \rightarrow b_1$ in (4.21) and $b \rightarrow b_2$ in (6.21).

Eliminating the auxiliary fields A_{ik} and F^A, \bar{F}_A from this sum by their equations of motion

$$\begin{aligned} A_{ik} &= i \left(\frac{1}{\alpha} + 2 \right) r^{-1} \varphi_{(i} \bar{\varphi}_{k)} - 4b_1 \alpha^2 i r^{\frac{1}{\alpha}+2} \phi_{(i}^A \phi_{k)A}, \\ F^A &= \left[b_1 r + b_2 (f^2)^{-1} \right]^{-1} \left[b_1 \varphi_k + 2b_2 (f^2)^{-2} f_{ka} \varphi^a \right] \phi^{kA}, \\ \bar{F}_A &= \left[b_1 r + b_2 (f^2)^{-1} \right]^{-1} \left[b_1 \bar{\varphi}_k + 2b_2 (f^2)^{-2} f_{ka} \bar{\varphi}^a \right] \phi_A^k, \end{aligned} \quad (7.1)$$

we obtain the action with the Lagrangian

$$\begin{aligned} L &= \frac{1}{8\alpha^2} r^{-\frac{1}{\alpha}-2} \left[\dot{r}\dot{r} + i \left(\bar{\varphi}_k \dot{\varphi}^k - \dot{\bar{\varphi}}_k \varphi^k \right) \right] + \frac{1}{2\alpha^2} (f^2)^{\frac{1}{\alpha}-1} \left[\dot{f}^{ia} \dot{f}_{ia} + \frac{i}{2} \left(\bar{\varphi}_a \dot{\varphi}^a - \dot{\bar{\varphi}}_a \varphi^a \right) \right] \\ &\quad - i \left[b_1 r + b_2 (f^2)^{-1} \right] \phi^{iA} \dot{\phi}_{iA} \\ &\quad - \frac{i}{\alpha^2} \left(\frac{1}{\alpha} - 1 \right) (f^2)^{\frac{1}{\alpha}-2} f^{ia} \dot{f}_i^b \varphi_{(a} \bar{\varphi}_{b)} - 2ib_2 (f^2)^{-2} f_i^a \dot{f}_{ka} \phi^{iA} \phi_A^k \\ &\quad + \frac{1+2\alpha}{48\alpha^4} r^{-\frac{1}{\alpha}-4} \varphi_{(i} \bar{\varphi}_{k)} \varphi^i \bar{\varphi}^k - \frac{1}{6\alpha^3} \left(\frac{1}{\alpha} - 1 \right) (f^2)^{\frac{1}{\alpha}-2} \varphi^a \bar{\varphi}^b \varphi_{(a} \bar{\varphi}_{b)} \\ &\quad - \frac{b_1}{2\alpha} r^{-1} \left[b_1 r + b_2 (f^2)^{-1} \right]^{-1} \left[b_1 r + b_2 (1+2\alpha)(f^2)^{-1} \right] \varphi_i \bar{\varphi}_k \phi^{iA} \phi_A^k \\ &\quad - 4b_1 b_2 r (f^2)^{-3} \left[b_1 r + b_2 (f^2)^{-1} \right]^{-1} f_{ia} f_{kb} \varphi^a \bar{\varphi}^b \phi^{iA} \phi_A^k \\ &\quad + 2b_1 b_2 (f^2)^{-2} \left[b_1 r + b_2 (f^2)^{-1} \right]^{-1} \left(\varphi_i \bar{\varphi}^a - \bar{\varphi}_i \varphi^a \right) f_{ka} \phi^{iA} \phi_A^k \\ &\quad + \alpha^2 (b_1)^2 r^{\frac{1}{\alpha}+2} \phi_{(i}^A \phi_{k)A} \phi^{iB} \phi_B^k. \end{aligned} \quad (7.2)$$

Redefining the variables as

$$x_1 = r^{-\frac{1}{2\alpha}}, \quad x_2 = (f^2)^{\frac{1}{2\alpha}}, \quad (7.3)$$

$$L_{ia} = \frac{1}{\alpha} (f^2)^{-\frac{1}{2}} f_{ia}, \quad L^{ia} L_{ia} = \frac{1}{\alpha^2}, \quad (7.4)$$

$$\psi_1^i = -\frac{1}{2\alpha} r^{-\frac{1}{2\alpha}-1} \varphi^i, \quad \bar{\psi}_{1i} = -\frac{1}{2\alpha} r^{-\frac{1}{2\alpha}-1} \bar{\varphi}_i, \quad (7.5)$$

$$\psi_2^i = \frac{1}{\alpha} (f^2)^{\frac{1}{2\alpha}-1} f^{ia} \varphi_a, \quad \bar{\psi}_{2i} = -\frac{1}{\alpha} (f^2)^{\frac{1}{2\alpha}-1} f_{ia} \bar{\varphi}^a, \quad (7.6)$$

$$\chi^{iA} = \sqrt{2} [b_1 r + b_2 (f^2)^{-1}]^{\frac{1}{2}} \phi^{iA}, \quad (7.7)$$

we bring the Lagrangian (7.2) into the form

$$\begin{aligned} L = & \frac{1}{2} \left[\dot{x}_1 \dot{x}_1 + \dot{x}_2 \dot{x}_2 + (x_2)^2 \dot{L}^{ia} \dot{L}_{ia} + i \left(\bar{\psi}_{1k} \dot{\psi}_1^k - \dot{\bar{\psi}}_{1k} \psi_1^k + \bar{\psi}_{2k} \dot{\psi}_2^k - \dot{\bar{\psi}}_{2k} \psi_2^k \right) - i \chi^{iA} \dot{\chi}_{iA} \right] \\ & + i \alpha L_i^a \dot{L}_{ka} \left\{ 2 \psi_2^{(i} \bar{\psi}_2^{k)} - \alpha b_2 (x_2)^{-2\alpha} \left[b_1 (x_1)^{-2\alpha} + b_2 (x_2)^{-2\alpha} \right]^{-1} \chi^{iA} \chi_A^k \right\} \\ & + \frac{1}{3} (1 + 2\alpha) (x_1)^{-2} \psi_{1(i} \bar{\psi}_{1k)} \psi_1^i \bar{\psi}_1^k - \frac{2}{3} (1 - \alpha) (x_2)^{-2} \psi_{2(i} \bar{\psi}_{2k)} \psi_2^i \bar{\psi}_2^k \\ & - \alpha b_1 (x_1)^{-2\alpha-2} \left[b_1 (x_1)^{-2\alpha} + b_2 (1 + 2\alpha) (x_2)^{-2\alpha} \right] \left[b_1 (x_1)^{-2\alpha} + b_2 (x_2)^{-2\alpha} \right]^{-2} \psi_{1i} \bar{\psi}_{1k} \chi^{iA} \chi_A^k \\ & - 2\alpha^2 b_1 b_2 (x_1)^{-2\alpha} (x_2)^{-2\alpha-2} \left[b_1 (x_1)^{-2\alpha} + b_2 (x_2)^{-2\alpha} \right]^{-2} \psi_{2i} \bar{\psi}_{2k} \chi^{iA} \chi_A^k \\ & + 2\alpha^2 b_1 b_2 (x_1)^{-2\alpha-1} (x_2)^{-2\alpha-1} \left[b_1 (x_1)^{-2\alpha} + b_2 (x_2)^{-2\alpha} \right]^{-2} \left[\psi_{1i} \bar{\psi}_{2k} - \psi_{2i} \bar{\psi}_{1k} \right] \chi^{iA} \chi_A^k \\ & + \frac{1}{4} \alpha^2 (b_1)^2 (x_1)^{-4\alpha-2} \left[b_1 (x_1)^{-2\alpha} + b_2 (x_2)^{-2\alpha} \right]^{-2} \chi_{(i}^A \chi_{k)A} \chi^{iB} \chi_B^k. \end{aligned} \quad (7.8)$$

The on-shell supersymmetry transformations leaving (7.8) invariant up to a total derivative are as follows

$$\delta x_1 = -\omega_i \psi_1^i + \bar{\omega}^i \bar{\psi}_{1i}, \quad \delta x_2 = -\omega_i \psi_2^i + \bar{\omega}^i \bar{\psi}_{2i}, \quad (7.9)$$

$$\delta L^{ia} = -2\alpha (x_2)^{-1} \left[\omega^{(i} \psi_2^{j)} + \bar{\omega}^{(i} \bar{\psi}_2^{j)} \right] L_j^a, \quad (7.10)$$

$$\begin{aligned} \delta \psi_1^i &= i \bar{\omega}^i \dot{x}_1 + i \bar{\eta}^i x_1 - (1 + 2\alpha) (x_1)^{-1} (\omega_k \psi_1^k \psi_1^i + \bar{\omega}_k \psi_1^k \bar{\psi}_1^i) \\ &\quad + \alpha b_1 (x_1)^{-2\alpha-1} \left[b_1 (x_1)^{-2\alpha} + b_2 (x_2)^{-2\alpha} \right]^{-1} \bar{\omega}_k \chi^{A(i} \chi_A^k), \\ \delta \bar{\psi}_{1i} &= -i \omega_i \dot{x}_1 - i \eta_i x_1 + (1 + 2\alpha) (x_1)^{-1} (\bar{\omega}^k \bar{\psi}_{1k} \bar{\psi}_{1i} + \omega^k \bar{\psi}_{1k} \psi_{1i}) \\ &\quad + \alpha b_1 (x_1)^{-2\alpha-1} \left[b_1 (x_1)^{-2\alpha} + b_2 (x_2)^{-2\alpha} \right]^{-1} \omega^k \chi_{(i}^A \chi_{k)A}, \end{aligned} \quad (7.11)$$

$$\begin{aligned} \delta \psi_2^i &= i \bar{\omega}^i \dot{x}_2 + i \bar{\eta}^i x_2 - 2\alpha i x_2 L^{ia} \dot{L}_a^k \bar{\omega}_k \\ &\quad - (1 + 2\alpha) (x_2)^{-1} \omega_k \psi_2^k \psi_2^i - 2\alpha (x_2)^{-1} \bar{\omega}_k \psi_2^k \bar{\psi}_2^i + (x_2)^{-1} \bar{\omega}_k \psi_2^i \bar{\psi}_2^k, \\ \delta \bar{\psi}_{2i} &= -i \omega_i \dot{x}_2 - i \eta_i x_2 - 2\alpha i x_2 L_i^a \dot{L}_{ka} \omega^k \\ &\quad + (1 + 2\alpha) (x_2)^{-1} \bar{\omega}^k \bar{\psi}_{2k} \bar{\psi}_{2i} + 2\alpha (x_2)^{-1} \omega^k \bar{\psi}_{2k} \psi_{2i} - (x_2)^{-1} \omega^k \bar{\psi}_{2i} \psi_{2k}, \end{aligned} \quad (7.12)$$

$$\begin{aligned} \delta \chi^{iA} = & -2\alpha \left[b_1 (x_1)^{-2\alpha} + b_2 (x_2)^{-2\alpha} \right]^{-1} \left\{ \omega^{(i} \left[b_1 (x_1)^{-2\alpha-1} \psi_1^{j)} + b_2 (x_2)^{-2\alpha-1} \psi_2^{j)} \right] \right. \\ & \left. + \bar{\omega}^{(i} \left[b_1 (x_1)^{-2\alpha-1} \bar{\psi}_1^{j)} + b_2 (x_2)^{-2\alpha-1} \bar{\psi}_2^{j)} \right] \right\} \chi_j^A. \end{aligned} \quad (7.13)$$

It is straightforward to compute the relevant $D(2, 1; \alpha)$ Noether (super)charges and Casimirs, as it was done in the previous sections for the separate pair couplings. We leave finding the explicit expressions for the future.

Let us dwell on some peculiarities of the system constructed.

First, the component action associated with (7.8) involves a non-trivial dependence on both coupling constants b_1 and b_2 , in contrast to the actions for the separate pairs which are reproduced by setting $b_1 = 0$ or $b_2 = 0$.

Accordingly, we observe that the quartic fermionic terms in (7.8) have a more complicated dependence on x_1 and x_2 as compared to the separate $(\mathbf{1}, \mathbf{4}, \mathbf{3}) \oplus (\mathbf{0}, \mathbf{4}, \mathbf{4})$ or $(\mathbf{4}, \mathbf{4}, \mathbf{0}) \oplus (\mathbf{0}, \mathbf{4}, \mathbf{4})$ couplings, in which cases we are left with the standard conformal denominators $(x_1)^{-2}$ or $(x_2)^{-2}$.

These peculiarities survive in the special cases $\alpha = -1/2$ corresponding to the superconformal group $OSp(2|4)$ and $\alpha = -1$ which corresponds to $PSU(1, 1|2)$ and was the subject of study in [27]. It would be interesting to reveal implications of this and, perhaps, some other “hybrid” $D(2, 1; \alpha)$ invariant $d = 1$ systems in the AdS/CFT and supersymmetric black hole business along the lines of Ref. [27] and related references.

8 Concluding remarks

In this paper we have presented in detail the construction of the realizations of the most general $\mathcal{N} = 4, d = 1$ superconformal symmetry $D(2, 1; \alpha)$ in the SCM models associated with the reducible $\mathcal{N} = 4$ multiplets $(\mathbf{1}, \mathbf{4}, \mathbf{3}) \oplus (\mathbf{0}, \mathbf{4}, \mathbf{4})$, $(\mathbf{3}, \mathbf{4}, \mathbf{1}) \oplus (\mathbf{0}, \mathbf{4}, \mathbf{4})$ and $(\mathbf{4}, \mathbf{4}, \mathbf{0}) \oplus (\mathbf{0}, \mathbf{4}, \mathbf{4})$. In all cases, we started from the manifestly supersymmetric off-shell superfield description, then passed to the off-shell component actions and, finally, to the on-shell actions by eliminating the relevant sets of the auxiliary fields. Though the superfield description for the separate multiplets entering the various pairs was known before, neither full off- and on-shell superconformal Lagrangians nor the explicit realizations of the $D(2, 1; \alpha)$ generators for them at arbitrary α were given. We also worked out an instructive example of the $D(2, 1; \alpha)$ invariant action involving the multiplets $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ and $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ interacting through couplings to the multiplet $(\mathbf{0}, \mathbf{4}, \mathbf{4})$. All the models constructed admit a simple extension to an arbitrary number of the multiplets $(\mathbf{0}, \mathbf{4}, \mathbf{4})$, like in the $\alpha = -1$ case treated in [27].

The common feature of all models considered is the splitting of the involved variables into the radial (“dilaton”) part presented by the field x and the angular part containing everything else, including the fermionic variables of the multiplets $(\mathbf{0}, \mathbf{4}, \mathbf{4})$. The structure of supercharges and Hamiltonians in terms of this splitting is basically universal for all considered systems, like in their $\alpha = -1$ particular case, and is, presumably, in the agreement with the general structure of $\mathcal{N} = 4$ SCM models with $D(2, 1; \alpha)$ invariance suggested in [19]. As distinct from [19], we begin with the well defined reducible off-shell $\mathcal{N} = 4, d = 1$ representation and come to the final expressions for the $D(2, 1; \alpha)$ generators and the splitting of variables just mentioned through the standard Noether procedure applied to the relevant invariant Lagrangians and by making the universal field redefinition in the end. Another difference is that our off-shell approach, as is illustrated in section 7, allows one to gain, as a result of elimination of the auxiliary fields, some additional fermionic couplings which are difficult to guess within the intrinsically on-shell Hamiltonian approach.

It is of interest to apply the same methods to construct more general superconformal

$\mathcal{N} = 4$ models, involving, e.g., some mirror (or “twisted”) analogs [33], [34] of the multiplets considered here and to generalize our study to the $D(2, 1; \alpha)$ invariant models with the “trigonometric” realization of the conformal $d = 1$ symmetry along the lines of Ref. [28]. It is also desirable to consider quantum versions of all these models and $D(2, 1; \alpha)$ realizations, as well as to understand in full generality their possible links with the higher-dimensional supergravity, black holes and AdS/CFT in the spirit of refs. [4], [6], [27], [26] and other related works.

Finally, we focus on the two surprising common features of the models presented.

One unusual property is seen already at the level when the extra $(\mathbf{0}, \mathbf{4}, \mathbf{4})$ multiplets are suppressed. In the three cases considered in sections 4, 5 and 6 the corresponding Lagrangians are (without WZ terms for the $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ multiplet)

$$L_{(1,4,3)} = \frac{1}{2} \left[\dot{x}\dot{x} + i \left(\bar{\psi}_k \dot{\psi}^k - \dot{\bar{\psi}}_k \psi^k \right) \right] + \frac{1}{3} (1 + 2\alpha) x^{-2} \psi^i \bar{\psi}^k \psi_{(i} \bar{\psi}_{k)}, \quad (8.1)$$

$$\begin{aligned} L_{(3,4,1)} &= \frac{1}{2} \left[\dot{x}\dot{x} + x^2 \dot{\ell}^{ik} \dot{\ell}_{ik} + i \left(\bar{\psi}_k \dot{\psi}^k - \dot{\bar{\psi}}_k \psi^k \right) \right] + 4i\alpha \ell_i^j \dot{\ell}_{kj} \psi^{(i} \bar{\psi}^{k)} \\ &\quad - \frac{1}{3} (1 - 2\alpha) x^{-2} \psi^i \bar{\psi}^k \psi_{(i} \bar{\psi}_{k)}, \end{aligned} \quad (8.2)$$

$$\begin{aligned} L_{(4,4,0)} &= \frac{1}{2} \left[\dot{x}\dot{x} + x^2 \dot{L}^{ia} \dot{L}_{ia} + i \left(\bar{\psi}_k \dot{\psi}^k - \dot{\bar{\psi}}_k \psi^k \right) \right] + 2i\alpha L_i^a \dot{L}_{ka} \psi^{(i} \bar{\psi}^{k)} \\ &\quad - \frac{2}{3} (1 - \alpha) x^{-2} \psi^i \bar{\psi}^k \psi_{(i} \bar{\psi}_{k)}, \end{aligned} \quad (8.3)$$

where x is the radial variable and ℓ_{ik} , L_{ia} are angular variables. We see, that the four-fermionic terms are vanishing at *different* values of α for different multiplets: $\alpha = -1/2$ for the multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$, $\alpha = 1/2$ for the multiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ and $\alpha = 1$ for the multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$.

On the other hand, in the supercharges

$$Q_{(1,4,3)}^i = p \psi^i - \frac{2i}{3} (1 + 2\alpha) x^{-1} \psi_k \psi^{(i} \bar{\psi}^{k)}, \quad (8.4)$$

$$Q_{(3,4,1)}^i = p \psi^i - \frac{2i}{3} (1 + 2\alpha) x^{-1} \psi_k \psi^{(i} \bar{\psi}^{k)} - 4\alpha x^{-1} \psi_k \ell_j^{(i} p^{k)j}, \quad (8.5)$$

$$Q_{(4,4,0)}^i = p \psi^i - \frac{2i}{3} (1 + 2\alpha) x^{-1} \psi_k \psi^{(i} \bar{\psi}^{k)} - 2\alpha x^{-1} \psi_k L^{(i} p^{k)a} \quad (8.6)$$

the three-fermion terms vanish at $\alpha = -1/2$ for *all* multiplets we dealt with. Moreover, the angular *dynamical* variables in the supercharges can be absorbed into the relevant $SU(2)$ currents.

The second common feature is that the contribution of the $(\mathbf{0}, \mathbf{4}, \mathbf{4})$ multiplets to the supercharges is universal for all multiplets:

$$Q_{(1,4,3) \oplus (0,4,4)}^i = p \psi^i - ix^{-1} \psi_k \left[\frac{2}{3} (1 + 2\alpha) \psi^{(i} \bar{\psi}^{k)} - \alpha \chi^{iA} \chi_A^k \right], \quad (8.7)$$

$$Q_{(3,4,1) \oplus (0,4,4)}^i = p \psi^i - ix^{-1} \psi_k \left[\frac{2}{3} (1 + 2\alpha) \psi^{(i} \bar{\psi}^{k)} - 4i\alpha \ell_j^{(i} p^{k)j} - \alpha \chi^{iA} \chi_A^k \right], \quad (8.8)$$

$$Q_{(4,4,0) \oplus (0,4,4)}^i = p \psi^i - ix^{-1} \psi_k \left[\frac{2}{3} (1 + 2\alpha) \psi^{(i} \bar{\psi}^{k)} - 2i\alpha L^{(i} p^{k)a} - \alpha \chi^{iA} \chi_A^k \right]. \quad (8.9)$$

It would be interesting to learn whether these properties survive quantization.

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